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On Finite Deformations of an Elastic Isotropic Material

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ON FINITE DEFORMATIONS OF AN ELASTIC ISOTROPIC MATERIAL

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Introduction

This report contains some results on finite deformations of an ideal elastic material.

The first chapter deals with a purely geometric question, which is quite independent of the physical properties of materials. The question is how much a 1-1 transformation of a domain D can differ from a rigid motion, if the maximum relative extension

$$\left|\frac{ds' - ds}{ds}\right| \tag{1}$$

occurring is at most ε. The answer to this question gives information on the size of the strains that are bound to occur in a solid when the solid is subjected to a deformation that changes the distance of one pair of points by a certain amount. It is obvious that for a solid of diameter $\triangle$ a deformation with maximum extension ε cannot increase the distance of any two points by more than $\varepsilon\triangle$. It is not so obvious that mutual distances between two points could not decrease by larger amounts. What distance changes are compatible with extensions of maximum size ε depends on the shape of the body. The geometric analysis is carried through here only for the case of a square plate of side a and thickness h, where $h \leq a$. It is proved that there exists a constant M with the following property. For any transformation $x' = f(x)$ of the plate (in vector notation) there exists a rigid motion $x' = g(x)$ [$g(x)$ is a linear transformation with constant coefficients] such that

$$|f(x) - g(x)| \leq M\varepsilon a^2/h \quad . \tag{2}$$

The best value for M is not determined here. It depends doubtlessly on the range of ε-values admitted. If, say, only values of ε below 1/100 are considered, better estimates for M can be obtained, than if we permit for ε all values between 0 and 1. For practical purposes it would seem worthwhile to obtain the best M for the limiting case $\varepsilon \longrightarrow 0$.

Formula (2) shows that a deformation of a plate differs from a rigid motion by a quantity of order εa, unless the thickness h of the plate is small in comparison to its width a. It is shown for thin plates the components of displacement tangential to the plate stay of order εa (at least for $\varepsilon \ll h^2/a^2$), while the normal components can be of order $\varepsilon a^2/h$.

The proof of (2) is based on the realization that there exist subsets called "cores" of the domain D which is undergoing the transformation, in which the relative change in distance of any two points is at most ε. In fact, any convex subset of D whose distance from the boundary of D exceeds a certain amount $\delta = \delta(\varepsilon, D)$ is such a core. Moreover, δ tends to 0 for $\varepsilon \to 0$. This implies that for convex D and small ε there are cores of D which fill all of D except for a thin boundary layer. Since in a core the mutual distance between any two points changes by a small amount for small ε, the transformation of a core is essentially rigid, which leads to the desired result. Similar estimates can doubtlessly be obtained for D which have shapes differing from square plates.

We have here measured the amount by which the transformation f differs from a rigid motion g by the maximum of f-g. It is more customary to measure the difference by the difference of the matrices of the first derivatives of f and g. Let p denote the matrix formed from all first derivatives of the components of f ("Jacobian matrix"). The matrix p can be decomposed into an orthogonal matrix c giving the local rotation and a symmetric matrix $1+\eta$ determining the strain:

$$cp = 1 + \eta \quad .$$

For small elongations η is small. For $\eta = 0$ we have a rigid motion and c must be a constant orthogonal matrix. The question then is how much c can differ from a constant matrix for small η.

It is shown by a counter example that small extensions do not guarantee that the local rotations at all points are approximately the same everywhere. However, this will be true <u>in the mean</u>. We can show that in the case of a square plate

subjected to a transformation with maximum extension ε there is a constant orthogonal matrix γ such that we have the estimate

$$\iiint (c - \gamma)(c - \gamma)^{*} dV = O(\varepsilon a^4/h)$$

for the norm of $c-\gamma$. This means that on the average p differs little from a constant orthogonal matrix for small ε. For h of the same order as a the norm is of order εa^3. [1]

Chapter II gives an exposition of the classical equations for ideal elastic isotropic materials. The relations are equivalent to those given by F. D. Murnaghan [2], Novozhilov [1] and by Rivlin [6]. The emphasis is however on arriving at a formulation that permits formulation and solution of problems purely in terms of Lagrange coordinates. The strains and stresses which are the quantities usually in the foreground are replaced here by their more immediate Lagrangian analogues

$$p_{ik} = \frac{\partial x_i'}{\partial x_k} \quad , \qquad q_{ik} = \frac{\partial U(p)}{\partial p_{ik}}$$

where U is the strain energy function. The advantage of working with these quantities (not sufficiently stressed in the literature) is that the equilibrium equations and compatibility conditions are the simple linear relations

$$q_{ik,k} = 0 \quad , \qquad p_{ik,r} = p_{ir,k} \quad ,$$

and that the boundary conditions are all taken on the known boundary of the body in the unstrained state. No curvature tensors have to be considered, and tensor calculus can be avoided entirely in favor of matrix computations. Non-linearity enters only through the formulae expressing the q_{ik} in terms of the p_{rs}.

The relations

$$q_{ik} = \frac{\partial U(p)}{\partial p_{ik}}$$

between p and q which take the place of the stress strain relations are analyzed in more detail in Chapter III. In

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[1] For a discussion of orders of magnitudes of rotations and extensions see Novozhilov [1].

particular, the expansion of q in terms of p near p = 1 is carried out up to terms of third order in p-1 since **information** on these terms is needed for the later discussion of buckling of plates.

In Chapter IV we consider changes in potential energy resulting from changes in the transformation $x' = f(x)$. The potential energy is stationary for the equilibrium state. It is shown that the equilibrium state actually corresponds to a minimum of potential energy if displacements are prescribed on the boundary, provided that for the deformations admitted the matrix p is restricted to a small finite neighborhood of the unit matrix. This yields a uniqueness theorem for the problem of determining the strained state of the body for given displacements on the boundary for an arbitrary choice of strain energy function.

Chapters V and VI derive the equations for bending of thin plates using Lagrange coordinates. In essence the method is similar to that used by Chien and Synge (see [3], [4]). The plate in the unstrained state is taken to be of constant thickness 2h with its faces given by $x_3 = \pm h$. It is assumed that the transformation corresponding to the strained state can be expanded into a formal power series in terms of x_3 and h. We obtain from the equilibrium equations, compatibility conditions and boundary conditions on the faces recursion formulae for the coefficients of the power series which are functions of x_1 and x_2. These recursion formulae essentially determine successively the coefficients and exclude a buckled solution unless on the limiting middle surface, i.e. for $x_3 = h = 0$ the determinant

$$\left| \frac{\partial^2 U(p)}{\partial p_{ik}\, \partial p_{rs}} \right|$$

vanishes. The determinant vanishes in particular when the stresses vanish to lowest order in h on the middle surface $x_3 = 0$. In that case the solution of the recursion formulae follows the pattern familiar from other asymptotic expansion

problems. At each stage of the expansion the recursion formulae determine the expansion coefficients only within an arbitrary function; differential equations for that arbitrary function are only found by considering the recursion formulae for the next higher orders and expressing that those are compatible with existence of a solution. In particular, the complete system of differential equations for the first non-vanishing quantities constitutes then what may be considered as <u>the differential equations for thin plates</u>.

In Chapter V this program is carried out for the case of two dimensional transformations, admitting large local rotations for $h \rightarrow 0$. The resulting differential equation for the inclination θ of the limiting middle surface is similar to the equation of the elastica. In Chapter VI three dimensional transformations are considered, in which however the middle surface shall in the limit for $h \rightarrow 0$ coincide with the original unstrained surface $x_3 = 0$ (instead of coinciding with a more general developable surface). The further assumption that the tangential stresses are of order h^2 leads to the equations of v.Karman and Föppl. It turns out that the differential equations for the lowest order non-vanishing quantities only involve two material constants, the Lamé constants λ and μ. In the derivation of the equations, higher order terms in the expansion of q in powers of p-1 have to be considered. These terms involve in addition to λ and μ seven further constants, which however for reasons of symmetry do not enter the final result.

Chapter I

Geometric Analysis of Deformations with Small Strain

Let a transformation be described by the equations

$$x_i' = f_i(x_1, x_2, x_3) \ , \qquad i = 1,2,3, \tag{1.1}$$

or in vector notation by

$$x' = f(x) \ . \tag{1.2}$$

We associate with the transformation the matrix

$$p = (p_{ik}) = \left(\frac{\partial x_i'}{\partial x_k}\right) \tag{1.3}$$

and assume that p has a positive determinant $|p|$. The corresponding local linear approximation to the transformation is then given by

$$dx' = p\,dx \ . \tag{1.4}$$

The <u>elongation</u> of a line element is represented by the expression

$$e = \frac{ds' - ds}{ds} = \sqrt{\frac{dx^* p^* p dx}{dx^* dx}} - 1 \tag{1.5}$$

where the asterisk denotes transposition.

The elongation in different directions is completely determined by the positive symmetric matrix

$$g = p^* p \ . \tag{1.6}$$

For dx varying in direction e is stationary when dx has the direction of an eigenvector of g. The stationary values of e are given by

$$\sqrt{\lambda} - 1 \tag{1.7}$$

where λ is an eigenvalue of g. Following K. O. Friedrichs we can introduce as <u>strain matrix</u> the matrix η determined by the conditions that $1+\eta$ is positive symmetric and that

$$(1.8) \qquad (1+\eta)^2 = g = p^*p \quad .$$

The principal elongations e are just the eigenvalues of η. If we introduce the matrix c by

$$(1.9) \qquad cp = 1+\eta$$

we have

$$p^*p = (1+\eta)^2 = (1+\eta^*)(1+\eta) = p^*c^*cp$$

and hence

$$c^*c = 1 \quad .$$

The matrix c will be orthogonal and of determinant +1, since p and $1+\eta$ have positive determinant. Equation (1.9) corresponds to the decomposition of the infinitesimal transformation into a rotation c^* and a pure dilatation represented by $1+\eta$:

$$p = c^*(1+\eta) \quad .$$

If the transformation is isometric we have

$$g = 1 \;, \quad e = 0 \;, \quad \eta = 0 \;, \quad p = c^* \quad .$$

In that case f must represent a rigid motion and the local rotation must be given by a matrix c^* with constant elements. The local rotation is then the same at all points. In this chapter we shall investigate what happens if one assumes that the elongation e is small for all line elements. We shall estimate how much the transformation f can differ from a rigid motion in this case.

Assume then that we apply a transformation $x' = f(x)$ with f of class C^1 to the points of a region R in x-space. Let there be given a fixed quantity ε such that for all x in R

$$(1.10) \qquad -\varepsilon \leq e = \frac{ds' - ds}{ds} \leq +\varepsilon$$

where $0 \leq \varepsilon < 1$.

To illustrate the behavior to be expected we consider two examples of transformations. The first example to be discussed will show that even though the elongation is small everywhere

the local rotation does not have to be approximately the same in all points. We make use of cylindrical coordinates r, θ, z:

$$x_1 = r\cos\theta\ , \quad x_2 = r\sin\theta\ , \quad x_3 = z$$

and consider a transformation of the form

$$r' = r\ , \quad \theta' = \theta - F(r)\ , \quad z' = z \tag{1.11}$$

with a suitable function F(r). The elements of length are given by

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$ds'^2 = ds^2 + r^2(d\theta'^2 - d\theta^2)\quad .$$

Here

$$|r^2(d\theta'^2 - d\theta^2)| = |rF'(-2r\,d\theta dr + rF'dr^2)|$$

$$\leq |rF'|(1 + |rF'|)ds^2 \leq \varepsilon ds^2$$

if

$$|rF'(r)| \leq \frac{\varepsilon}{2}\quad . \tag{1.12}$$

It follows that the elongation e satisfies (1.10), if (1.12) holds.

The matrix p is here given by

$$p = \begin{pmatrix} \cos F + rF'\sin\theta'\cos\theta & \sin F + rF'\sin\theta'\sin\theta & 0 \\ -\sin F - rF'\cos\theta'\cos\theta & \cos F - rF'\cos\theta'\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Taking for c the orthogonal matrix

$$c = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ +\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the matrix cp will be symmetric, when

$$\phi = F(r) + \arc \tan \left(\tfrac{1}{2} rF'(r)\right) .$$

If we now choose for F(r) the function

$$F(r) = \varepsilon \log \left(\log \frac{1}{r}\right) ,$$

condition (1.12) will be satisfied for $r < e^{-2}$, while

$$\lim_{r \to 0} \phi = \infty ,$$

so that the rotation angle ϕ takes every value in a neighborhood of the origin.

This shows that arbitrarily large variations in local rotation are compatible with small strains.

The next example indicates the magnitudes of the displacements in various directions to be expected in a transformation of a rectangular plate when the maximum elongation is of order ε. Let the plate be given by

$$(1.13) \qquad 0 \leq x_1 \leq a , \quad 0 \leq x_2 \leq a , \quad 0 \leq x_3 \leq h \leq a .$$

We apply the transformation

$$x_1' = (R-x_3)\sin(x_1/R) , \quad x_2' = x_2 , \quad x_3' = R - (R-x_3)\cos(x_1/R)$$

which takes the plate into a portion of a cylindrical shell. Here the parameter R shall have a value exceeding h and a. The strain matrix has the simple form

$$\eta = \begin{pmatrix} -x_3/R & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in this case, so that the maximum elongation is $\varepsilon = h/R$. The maximum horizontal displacement occurs for $x_1 = a$, $x_3 = h$ and has the value

$$(1.12a) \qquad \max |x_1' - x_1| = a - (R-h)\sin(a/R) = a\varepsilon(1+O(a^2\varepsilon/h^2)) .$$

The maximum vertical displacement occurs for $x_1 = a$, $x_3 = 0$ and has the value

$$(1.12b) \qquad \max |x_3' - x_3| = 2R \sin^2(a/2R) = O(a^2\varepsilon/h) \quad .$$

Since here by assumption $a\varepsilon/h = a/R < 1$ the maximum total displacement of any point is of order $a^2\varepsilon/h$. In particular, if h is of the same order as a the maximum displacement is of order εa. If h is small compared to a but $a^2\varepsilon/h^2$ is still bounded the horizontal displacement is still of order εa, whereas the vertical displacement goes up to the order $(a/h)\varepsilon a$. This latter situation arises when the total displacement of any point does not exceed the thickness h of the plate.

We find then in this example that for a rectangular plate with sides of the same order a and maximum elongation ε the displacement is of order εa. The horizontal displacement is still of the same order as long as $\sqrt{\varepsilon}$ does not exceed the ratio of thickness h to width a. In any case the total displacement is of order $\varepsilon a^2/h$, as long as ε does not exceed h/a. We shall find essentially the same result for general transformations of the plate with maximum elongation ε (after applying a suitable rigid motion to the transformed region).

Let then

$$x' = f(x)$$

be a transformation of a region R with boundary B, where

$$1 - \varepsilon \leq \frac{ds'}{ds} \leq 1 + \varepsilon$$

for $x \in R$. If C is any curve in R and C' its image under f, we have the inequalities

$$(1 - \varepsilon) \int_C ds \leq \int_{C'} ds' \leq (1 + \varepsilon) \int_C ds$$

for the lengths of the curves C and C'.

Let P, Q be any two points of R such that the line segment C with endpoints P, Q belongs to R. Let $\overline{PQ}$ denote the distance of P and Q. Then

$$\overline{P'Q'} \leq \int_{C'} ds' \leq (1+\varepsilon) \int_C ds = (1+\varepsilon)\,\overline{PQ} \quad .$$

If in addition the line segment γ' with endpoints P', Q' belongs to the image region R', then γ' is the image of an arc γ joining P and Q, and

$$\overline{P'Q'} = \int_{\gamma'} ds' \geq (1-\varepsilon) \int_{\gamma} ds \geq (1-\varepsilon)\, \overline{PQ} \quad .$$

Hence

$$(1.3) \qquad (1-\varepsilon)\overline{PQ} \leq \overline{P'Q'} \leq (1+\varepsilon)\overline{PQ} \quad ,$$

if P and Q are such that the segment PQ belongs to R, the segment $P'Q'$ to R'.

Consider now the case of a line segment PQ lying in R which is such that the segment $P'Q'$ does not lie in R'. Let S be a point on PQ such that $P'S'$ belongs to R'. The set of these points is closed; it also contains P but not Q. There is then a point S_o in this set furthest away from P. The segment $P'S_o'$ belongs then to R' but cannot lie completely in the interior of R'. Thus $P'S_o'$ contains a point T' of the boundary B' of R'. T' is the image of a point T on the boundary B of R. Let γ denote the original in R of the segment γ' joining P' and S_o'. Since γ passes through T we have (see Figure 1)

$$(1-\varepsilon)(\overline{PT} + \overline{TS_o}) \leq (1-\varepsilon) \int_{\gamma} ds \leq \int_{\gamma'} ds' = \overline{P'S_o'} \leq (1+\varepsilon)\overline{PS_o} \quad .$$

Then also

$$(1-\varepsilon)(\overline{PT} + \overline{TQ}) \leq (1-\varepsilon)(\overline{PT} + \overline{TS_o} + \overline{S_oQ}) \leq (1+\varepsilon)\overline{PS_o} + (1-\varepsilon)\overline{S_oQ}$$

$$\leq (1+\varepsilon)(\overline{PS_o} + \overline{S_oQ}) = (1+\varepsilon)\overline{PQ} \quad .$$

Hence there must exist a point T on the boundary B such that

$$\overline{PT} + \overline{TQ} \leq \frac{1+\varepsilon}{1-\varepsilon}\, \overline{PQ} \quad .$$

The points T satisfying this inequality fill a certain ellipse with foci P, Q and minor semi-axis

$$\frac{\sqrt{\varepsilon}}{1-\varepsilon}\, \overline{PQ} \quad .$$

This is also the largest distance any point P of the ellipse can have from the segment $\overline{PQ}$. It follows that for a segment PQ in R the inequality (11) holds, if every point of the segment PQ has a distance from the boundary B which exceeds

$$\frac{\sqrt{\varepsilon}}{1-\varepsilon}\,\overline{PQ} .$$

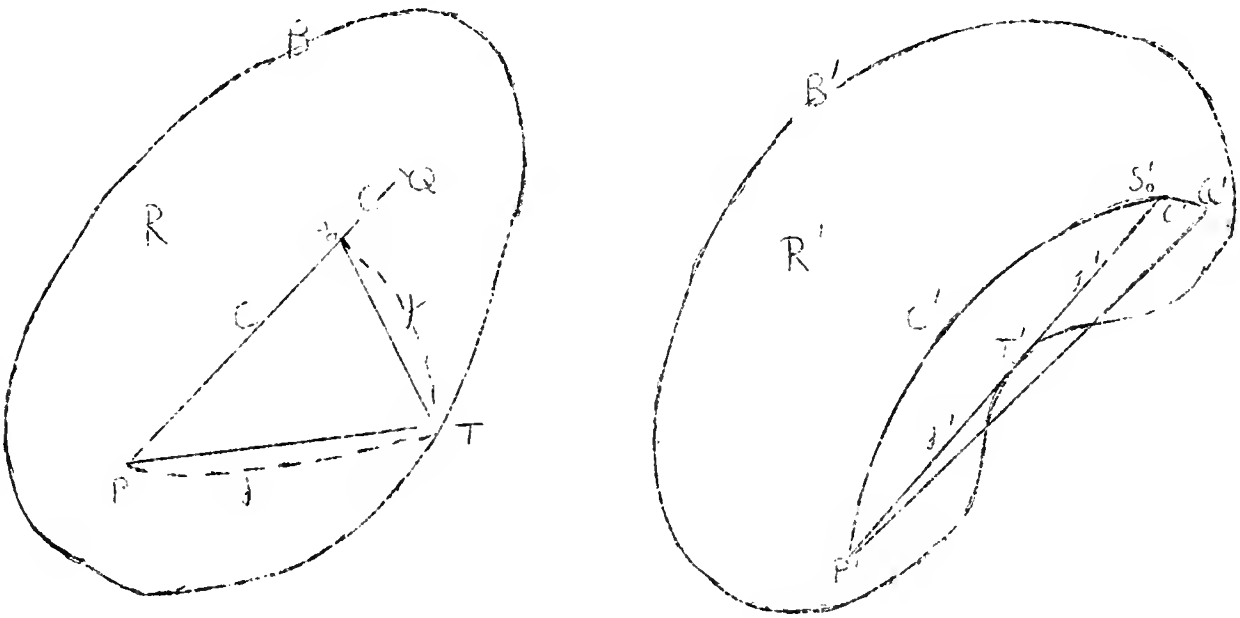

Fig. 1

We call a subregion D of R a <u>core</u>, if D is convex and if the convex hull of the image D' of D is contained in R'. If D is a core then the inequalities (11) hold for any two points of D. We have from the preceding argument that any convex subset D of R is a core if the minimum distance of D from the boundary B of R is at least

$$\frac{\sqrt{\varepsilon}}{1-\varepsilon}\,\Delta ,$$

where Δ is the diameter of D.

It will be seen that the transformation f can be approximated by a rigid motion in the points of a core. Let there be given a core that contains 4 points A_o, A_1, A_2, A_3 where the directions $\overrightarrow{A_oA_i}$ for i = 1,2,3 are mutually perpendicular. Let $a_i = \overline{A_oA_i}$. We make the assumption that

(1.14) $$\alpha_1 = \alpha_2 \geq \alpha_3 > 0 \ .$$

In a suitable rectangular coordinate system the A_i will have coordinates

$$A_o = (0,0,0), \quad A_1 = (\alpha_1,0,0), \quad A_2 = (0,\alpha_2,0), \quad A_3 = (0,0,\alpha_3) \ .$$

By a suitable orthogonal transformation applied to R' we can bring about that

$$A'_o = (0,0,0), \quad A'_1 = (a_{11},0,0), \quad A'_2 = (a_{21},a_{22},0), \quad A'_3 = (a_{31},a_{32},a_{33})$$

where a_{11}, a_{22}, a_{33} are positive.

For any point P of the core we put

$$r_i = \overline{A_i P} \ , \quad r'_i = \overline{A_i P'} \qquad \text{for} \quad i = 0,1,2,3 \ .$$

Then for $P = (x_1,x_2,x_3)$, $P' = (x'_1,x'_2,x'_3)$, $i = 1,2,3$

$$2\alpha_i x_i = r_o^2 - r_i^2 + \alpha_i^2 \ , \qquad 2\alpha_i x'_i = r_o'^2 - r_i'^2 + \alpha_i^2$$

and hence

(1.15) $$x'_i - x_i = \frac{(r'_o - r_o)(r'_o + r_o) - (r'_i - r_i)(r'_i + r_i)}{2\alpha_i} \ .$$

Here for $i = 0,1,2,3$

(1.16) $$r'_i = \overline{A_i P'} = \overline{A'_i P'} + O(\overline{A_i A'_i}) = \overline{A_i P} + O(\varepsilon \overline{A_i P}) + O(\overline{A_i A'_i})$$
$$= r_i + O(\varepsilon r_i) + O(\overline{A_i A'_i}) \ .$$

In particular for $i = 0$

(1.17) $$r'_o = r_o + O(\varepsilon r_o) \ .$$

If we take here for P the point A_1 relation (1.16) becomes

$$a_{11} = \alpha_1 + O(\varepsilon \alpha_1)$$

and hence

$$\overline{A_1 A'_1} = |a_{11} - \alpha_1| = O(\varepsilon \alpha_1) \ .$$

We restrict P to points of the core with $r_o \leq 4\alpha_1$. Then also

$$r_i = \overline{A_iP} \leq \overline{A_iA_o} + \overline{A_oP} = \alpha_i + r_o \leq 5\alpha_1 \quad .$$

We have then from (1.16) for $i = 1$

$$r_1' = r_1 + O(\varepsilon r_1) + O(\varepsilon\alpha_1) = r_1 + O(\varepsilon\alpha_1) = O(\alpha_1) \quad .$$

It follows from (1.15) that

$$x_1' - x_1 = O(\varepsilon\alpha_1) \quad . \tag{1.18}$$

In particular for $P = A_2, A_3$

$$a_{21} = O(\varepsilon\alpha_1) \;, \qquad a_{31} = O(\varepsilon\alpha_1) \quad . \tag{1.19}$$

Taking $P = A_2$ we have from (1.17), (1.19)

$$a_{22} = \sqrt{a_{22}^2 + a_{21}^2} + O(a_{21}) = r_o' + O(\varepsilon\alpha_1)$$

$$= r_o + O(\varepsilon\alpha_1) = \alpha_2 + O(\varepsilon\alpha_1)$$

and hence

$$\overline{A_2A_2'} = \sqrt{a_{21}^2 + (a_{22} - \alpha_2)^2} = O(\varepsilon\alpha_1) \quad .$$

Then generally for points P of the core with $r_o \leq 4\alpha_1$ by (1.16) and (1.15) (using $\alpha_2 = \alpha_1$)

$$r_2' = r_2 + O(\varepsilon\alpha_1) = O(\alpha_1)$$

$$x_2' - x_2 = O(\varepsilon\alpha_1) \quad . \tag{1.20}$$

In particular for $P = A_3$ relation (1.20) becomes

$$a_{32} = O(\varepsilon\alpha_1) \quad ,$$

and hence also

$$a_{33} = \sqrt{a_{33}^2 + a_{31}^2 + a_{32}^2} + O\left(\sqrt{a_{31}^2 + a_{32}^2}\right) = r_o' + O(\varepsilon\alpha_1)$$

$$= r_o + O(\varepsilon\alpha_1) = \alpha_3 + O(\varepsilon\alpha_1) \quad .$$

It follows that

$$\overline{A_3A_3'} = \sqrt{a_{31}^2 + a_{32}^2 + (a_{33} - \alpha_3)^2} = O(\varepsilon\alpha_1) \quad .$$

Then from (1.15), (1.16) for any P of the core with $r_o < 4\alpha_1$

$$r_3' = r_3 + O(\varepsilon\alpha_1) = O(\alpha_1)$$

(1.21) $$x_3' - x_3 = O(\varepsilon\alpha_1^2/\alpha_3) \quad .$$

Take now the case where R is the rectangular plate

(1.21a) $$0 \leq x_1 \leq a \ , \quad 0 \leq x_2 \leq a \ , \quad 0 \leq x_3 \leq h$$

with $0 < h \leq a$. For given positive δ the cell R_δ

(1.22) $$\delta \leq x_1 \leq a-\delta \ , \quad \delta \leq x_2 \leq a-\delta \ , \quad \delta \leq x_3 \leq h-\delta$$

will have distance δ from the boundary B of R, and its diameter will be at most $\sqrt{3}\,a$. It will be a core, if

$$\frac{\sqrt{3\varepsilon}}{1-\varepsilon}\, a \leq \delta \quad .$$

Assume that ε is so small that

(1.23) $$\varepsilon \frac{a^2}{h^2} \leq \frac{1}{48}(1-\varepsilon)^2$$

and take

(1.24) $$\delta = \frac{\sqrt{3\varepsilon}}{1-\varepsilon}\, a \quad .$$

Then $\delta \leq h/4$. The cell R_δ has sides

$$\alpha_1 = a - 2\delta \ , \quad \alpha_2 = a - 2\delta \ , \quad \alpha_3 = h - 2\delta \ ,$$

and hence is not empty. Moreover

$$\alpha_1 = O(a) \quad , \quad \alpha_1^2/\alpha_3 = O(a^2/h) \quad .$$

It follows from (1.18), (1.20), (1.21) that after a suitable orthogonal transformation applied to R' we have at least for the points of the core R_δ

$$(1.25)\qquad x_1' - x_1 = O(\varepsilon a)\ ,\quad x_2' - x_2 = O(\varepsilon a)\ ,\quad x_3' - x_3 = O(\varepsilon a^2/h)\ .$$

These are exactly the orders of magnitude for horizontal and vertical displacements that occurred in the special example (1.12 a,b) when $\varepsilon a^2/h$ was bounded. They are proved here only for the core, which however in the limit for $\varepsilon \to 0$ will fill the whole region R.

A different kind of argument is needed to estimate the displacements outside the core R_δ and also to treat the case where h is so small that (1.23) no longer holds. For this purpose we consider first the case of a core which contains a cube of side α and vertices $A_0, A_1, \ldots$. If P is a point of the core with $\overline{A_0P} = \alpha$ we have by (1.18), (1.20), (1.21) after a suitable rigid motion applied to R'

$$\overline{PP'} = O(\varepsilon\alpha)\ .$$

If θ denotes the angle PA_0P' we have

$$\measuredangle\, PA_0P' = \theta = O(PP'/A_0P) = O(\varepsilon)\ .$$

If Q is the point of distance α from A_0 on the ray from A_0 opposite to P and if Q also belongs to the core, then

$$\measuredangle\, QA_0Q' = O(\varepsilon)$$

and hence

$$\measuredangle\, P'A_0Q' = \measuredangle\, P'A_0'Q' = \pi + O(\varepsilon)\ .$$

Thus if PQ is a line segment of length 2α and A_0 its midpoint we have

$$\measuredangle\, P'A_0'A' = \pi + O(\varepsilon)\ ,$$

provided there exists a core that contains P, Q and also contains a cube of side α and vertex A_0. This is true regardless of the special rigid motion applied to R' which would not change the angle.

Let $P_0, P_1, \ldots, P_n$ be equidistant points on a line segment $L = P_0P_n$ of R, where

$$\overline{P_iP_{i+1}} = \alpha\ .$$

Let there exist for each $i = 1,2,\ldots,n-1$ a core containing P_{i-1} and P_{i+1} and also containing a cube of side α and vertex P_i. Let similarly P_o, P_1 belong to a cubical core of side α and vertex P_o. The angle between two successive vectors $\overrightarrow{P_i'P_{i+1}'}$ is then $O(\varepsilon)$. Hence the angle between any vector $\overrightarrow{P_i'P_{i+1}'}$ and $\overrightarrow{P_o'P_1'}$ is $O(n\varepsilon)$. We assumed that there was a cubical core of side α and vertex P_o. Let A_1, A_2, A_3 be the vertices of the core adjacent to P_o. We take P_o as origin and the adjacent edges of the cube as coordinate axes. By a suitable rigid motion we bring about that $P_o' = P_o$, A_1' lies on P_oA_1, A_2' lies in the plane $P_oA_1A_2$. Then the angle between $\overrightarrow{P_oP_1'}$ and $\overrightarrow{P_oP_1}$, that is the angle between $\overrightarrow{P_o'P_1'}$ and L is $O(\varepsilon)$. It follows that the angle between $\overrightarrow{P_i'P_{i+1}'}$ and L or between $\overrightarrow{P_i'P_{i+1}'}$ and $\overrightarrow{P_iP_{i+1}}$ is $O(n\varepsilon)$. Since also

$$\overline{P_i'P_{i+1}'} = \overline{P_iP_{i+1}} + O(\varepsilon\alpha) \quad ,$$

we have for $P_i = (x_{i1},x_{i2},x_{i3})$, $P_i' = (x_{i1}',x_{i2}',x_{i3}')$

$$x_{i+1\,k}' - x_{ik}' = x_{i+1\,k} - x_{ik} + O(n\varepsilon\alpha) \quad \text{for} \quad k = 1,2,3 \ .$$

Hence

$$x_{nk}' = \sum_{i=0}^{n-1} (x_{i+1\,k}' - x_{ik}') = \sum_{i=0}^{n-1} (x_{i+1\,k} - x_{ik}) + O(n^2\varepsilon\alpha)$$

$$= x_{nk} + O(n^2\varepsilon\alpha) \quad .$$

Consequently

$$\overline{P_n'P_n} = O(n^2\varepsilon\alpha) \quad .$$

Let again R be the region

$$0 \leq x_1 \leq a \ , \quad 0 \leq x_2 \leq a \ , \quad 0 \leq x_3 \leq h \leq a \quad .$$

Let R_{δ_1} be the subset of R described by

$$\delta_1 \leq x_1 \leq a-\delta_1 \ , \quad \delta_1 \leq x_2 \leq a-\delta_1 \ , \quad \delta_1 \leq x_3 \leq h-\delta_1 \quad ,$$

where

$$\delta_1 = \sqrt{\varepsilon}\, h \quad .$$

Then $\delta_1 \leq h/4$ if we assume that $\varepsilon < 1/16$. Take for P_o the point $(\delta_1, \delta_1, \delta_1)$ and for Q any point of R_{δ_1}. We divide the line segment $L = P_oQ$ into n equal parts α by points $P_1, P_2, \ldots, P_{n-1}$ with $P_n = Q$. We take here for n some integer satisfying

$$20\,\frac{a}{h} \leq n \leq 20\,\frac{a}{h} + 1 \quad .$$

Then

$$\alpha = \frac{1}{n}\,\overline{P_oQ} \leq \frac{1}{n}\sqrt{3}\,a \leq \frac{\sqrt{3}}{20}\,h \leq \frac{1}{2}(h - 2\delta_1) \quad .$$

Every point P_i of L is the vertex of a cube in R_{δ_1} with sides of length α which are parallel to the coordinate axes. The convex hull of this cube and of P_{i-1} and P_{i+1} is a core. For its diameter $\triangle$ is at most

$$(1 + \sqrt{3})\alpha \leq \frac{\sqrt{3}}{20}\,(1 + \sqrt{3})h$$

and its distance from B at least δ_1, so that

$$\frac{\sqrt{\varepsilon}}{1-\varepsilon}\,\triangle \leq \delta_1 \quad .$$

Consequently (after the proper orthogonal transformation applied to R')

$$\overline{QQ'} = O(n^2\varepsilon\alpha) = O(n\varepsilon\overline{P_oQ}) = O(n\varepsilon a) = O(\varepsilon a^2/h)$$

for any point Q of R_{δ_1}.

It remains to extend the inequality

$$\overline{QQ'} = O(\varepsilon a^2/h)$$

to all points of R.

If Q is any point of R there exists a point P in R_δ such that

$$\overline{PQ} \leq \delta \quad .$$

Then

$$\overline{Q'Q} \leq \overline{Q'P'} + \overline{P'P} + \overline{PQ} \leq (1 + \varepsilon)\overline{PQ} + \overline{P'P} + \overline{PQ} \leq (2 + \varepsilon)\delta + \overline{P'P} \quad .$$

If here the inequality

$$\overline{P'P} = O(\varepsilon a^2/h)$$

has been established already for the points P of R_δ, and if $\delta = O(\varepsilon a^2/h)$ it follows that

$$\overline{Q'Q} = O(\varepsilon a^2/h)$$

for the arbitrary point Q of R.

Taking here

$$\delta = \delta_1 = \sqrt{\varepsilon}\, h$$

we find that (1.26) holds generally in R, if

$$\sqrt{\varepsilon}\, h = O(\varepsilon a^2/h) \quad .$$

This is the case when

$$\sqrt{\varepsilon}\, \frac{a^2}{h^2} \geq 1 \quad .$$

Take on the other hand the case where

$$\sqrt{\varepsilon}\, \frac{a^2}{h^2} \leq 1 \quad . \tag{1.27}$$

For every Q in R there exists a cube C of side $2\delta_1$ which has Q as a vertex and sides parallel to the coordinate axes. Let the distance δ_2 of Q from the boundary B of R satisfy

$$\frac{\sqrt{12\ \varepsilon}}{1-\varepsilon}\, \delta_1 \leq \delta_2 \leq \delta_1 \quad .$$

The vertex A_o of C opposite to Q will have distance at least $2\delta_1$ from B and will lie inside R_{δ_1}. In fact the subcube C_1 of side δ_1 and vertex A_o will lie completely in R_{δ_1}.

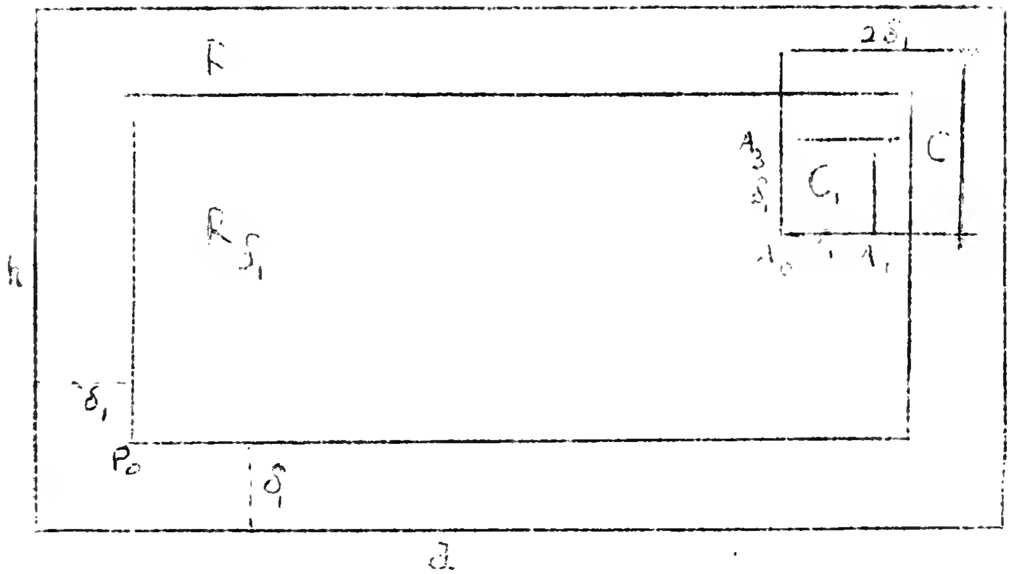

Fig. 2

Since C has diameter $\triangle = \sqrt{12}\,\delta_1$ and distance δ_2 from B the cube C is a core since

$$\frac{\sqrt{\varepsilon}}{1-\varepsilon} \triangle = \frac{\sqrt{12\,\varepsilon}}{1-\varepsilon}\,\delta_1 \leq \delta_2 \quad .$$

Let A_1, A_2, A_3 be the vertices of the subcube C_1 adjacent to A_o. Put

$$r_i = \overline{A_i Q}\ , \quad r_i' = \overline{A_i Q'} \qquad (i = 0,2,3) \ .$$

Then, since the A_i belong to R_{δ_1}

$$r_i = O(\delta_1)$$

$$r_i' = \overline{A_i Q'} = \overline{A_i' Q'} + O(\overline{A_i' A_i}) = r_i + O(\varepsilon r_i) + O(\overline{A_i' A_i})$$

$$= r_i + O(\varepsilon\delta_1) + O(\varepsilon a^2/h) = r_i + O(\varepsilon^{3/2} h) + O(\varepsilon a^2/h)$$

$$= r_i + O(\varepsilon a^2/h) \quad .$$

It follows from (1.27) that

$$\frac{r_i'^2 - r_i^2}{\delta_1} = O(\varepsilon a^2 h^{-1}(1 + \varepsilon a^2 h^{-1}\delta_1^{-1}))$$

$$= O(\varepsilon a^2 h^{-1}(1 + \varepsilon^{1/2} a^2 h^{-2})) = O(\varepsilon a^2/h) \quad .$$

Applying equation (1.15) with $\alpha_1 = \alpha_2 = \alpha_3 = \delta_1$ we find that for the coordinates x_i of Q and x_i' of Q'

$$x_i' - x_i = O(\varepsilon a^2/h)$$

and hence that

$$\overline{Q'Q} = O(\varepsilon a^2/h)$$

for any Q in R_{δ_3}, where

$$\delta_3 = \frac{\sqrt{12\,\varepsilon}}{1-\varepsilon}\,\delta_1 = \frac{\sqrt{12\,\varepsilon}}{1-\varepsilon}\,h \quad .$$

Since

$$\delta_3 = O(\varepsilon h) = O(\varepsilon a^2/h)$$

it follows then that (1.26) holds for any Q in R (see page 19).

Summarizing we have the following theorem:

There exists a constant M with the following property: For any transformation $x' = f(x)$ with maximum elongation ε applied to a square plate with sides a, a, h (where $h \leq a$) the maximum displacement of any point after applying a suitable orthogonal transformation is at most

$$M\varepsilon a^2/h \quad .$$

We can conclude that <u>on the average</u> the matrix p will differ little from the unit matrix and that the local rotation matrix c will also differ little from the identity.

First of all the symmetric matrix η can be brought into diagonal form by an orthogonal transformation. Since the eigenvalues of η are at most ε in absolute value and the orthogonal matrix has bounded elements, it follows that $\eta = O(\varepsilon)$. Then

$$p = c^*(1+\eta) = O(1+\varepsilon) = O(1) \quad .$$

Consider now any parallel to the x_1-axis. On it $ds = dx_1$ and

$$dx_i' = p_{i1}dx_1 = p_{i1}ds \quad , \qquad \frac{ds'}{ds} = \sqrt{p_{11}^2 + p_{21}^2 + p_{31}^2} \quad .$$

Since ds'/ds lies between $1-\varepsilon$ and $1+\varepsilon$ we have

$$(1-\varepsilon)^2 \leq p_{11}^2 + p_{21}^2 + p_{31}^2 \leq (1+\varepsilon)^2 \quad .$$

In particular,

$$|p_{11}| \leq 1+\varepsilon \quad .$$

Integrating over the portion of the parallel to the x_1-axis lying inside R we have

$$\int dx_1' = \int dx_1 + O(\varepsilon(a^2/h)) \quad ,$$

since the endpoints have moved at most by amounts $O(\varepsilon(a^2/h))$. This relation can be written

$$\int p_{11} \, dx_1 = a + O(\varepsilon(a^2/h))$$

or

$$\int (1+\varepsilon - p_{11}) dx_1 = O(\varepsilon(a^2/h) \quad .$$

Integrating over all parallels to the x_1-axis we find

$$\iiint_R (1+\varepsilon - p_{11}) dx = O(\varepsilon a^3) \qquad (dx = dx_1 \, dx_2 \, dx_3) \; .$$

Here the integrand is non-negative. Since

$$|1 - p_{11}| \leq |1+\varepsilon - p_{11}| + |\varepsilon| = (1+\varepsilon - p_{11}) + \varepsilon$$

it follows that

$$\iiint_R |1 - p_{11}| dx = O(\varepsilon a^3)$$

p_{11} being bounded, we have

$$(1 - p_{11})^2 = O(|1 - p_{11}|)$$

and thus also

$$\iiint_R (1 - p_{11})^2 dx = O(\varepsilon a^3) \quad .$$

We see that in the mean p_{11} differs little from 1. Moreover,

$$p_{21}^2 + p_{31}^2 \leq (1+\varepsilon - p_{11})(1+\varepsilon + p_{11}) = O(1+\varepsilon - p_{11}) .$$

Hence
$$\iiint_R \left[(p_{11}-1)^2 + p_{21}^2 + p_{31}^2\right] dx = O(\varepsilon a^3) .$$

In the same manner one finds that

$$\iiint_R \left[p_{12}^2 + (p_{22}-1)^2 + p_{32}^2\right] dx = O(\varepsilon a^3) ,$$

$$\iiint_R \left[p_{13}^2 + p_{23}^2 + (p_{33}-1)^2\right] dx = O(\varepsilon (a^4/h)) .$$

Altogether
$$\iiint_R (p-1)(p^*-1) dx = O(\varepsilon (a^4/h)) .$$

We measure the deviation of the rotation matrix c from unity by the matrix

$$(c^*-1)(c-1) .$$

Now
$$c = (1+\eta)p^{-1} = (1+\eta)^{-1}p^* = p^* - \eta(1+\eta)^{-1}p^* = p^* + O(\varepsilon) .$$

Hence
$$\iiint_R (c^*-1)(c-1) dx = \iiint_R (p-1)(p^*-1) dx + O(\varepsilon a^3) = O(\varepsilon (a^4/h)) .$$

This shows that for small ε the rotation matrix c differs in the mean little from a constant matrix.

Chapter II

Equations of Motion and Equilibrium in Lagrange Coordinates

A material is considered here in two states, "strained" and "unstrained". The Cartesian coordinates of a particle in the unstrained state ("Lagrange coordinates") shall be $(x_1,x_2,x_3) = x$. The Cartesian coordinates of the same particle in the strained state ("Euler coordinates") at the time t shall be $(x_1',x_2',x_3') = x'$, where

$$x' = x'(x,t) \quad .$$

We introduce again the Jacobian matrix

$$p = (p_{ik}) = \left(\frac{\partial x_i'}{\partial x_k}\right) \quad . \tag{2.1}$$

Velocity and acceleration are respectively the vectors with components

$$\dot{x}_i = \frac{\partial x_i'}{\partial t} \quad , \quad \ddot{x}_i' = \frac{\partial^2 x_i'}{\partial t^2} \quad .$$

Let dV and ρ denote element of volume and density in the unstrained state, dV' and ρ' that in the strained state. Conservation of mass implies that for the element of mass

$$dm = \rho\, dV = \rho' dV' \quad , \tag{2.2}$$

where

$$dV' = |p|\, dV \quad . \tag{2.3}$$

(Here $|p|$ denotes the determinant of p, which is assumed to be positive.)

The stress in the strained state is described by a matrix $\tau = (\tau_{ik})$, such that the force acting on a surface element dS' with unit normal ξ' in the strained state is given by

$$\tau_{ik}\xi_k' dS' \quad . \tag{2.4}$$

In addition to the stresses there may be present external forces represented by a vector $F = (F_1,F_2,F_3)$, such that $F dV'$ represents the external force acting on a volume element dV' in the strained state.

Let R' be a region with boundary S' cut out from the material in the strained state. Let ξ_i' be the components of the exterior unit normal of S'. Then Newton's law of motion gives

$$\iiint_{R'} \rho' \ddot{x}_i' \, dV' = \iiint_{R'} F_i \, dV' + \iint_{S'} \tau_{ik} \xi_k' \, dS' \ .$$

Applying the divergence theorem to the surface integral and shrinking R' into a point we find the equations of motion

$$\rho' \ddot{x}_i' = F_i + \frac{\partial \tau_{ik}}{\partial x_k'} \ . \tag{2.5}$$

The work done by the surface stresses and exterior forces on R' in unit time is

$$W = \iint_{S'} \tau_{ik} \xi_k' \dot{x}_i' \, dS' + \iiint_{R'} F_i \dot{x}_i' \, dV' \ . \tag{2.6}$$

The kinetic energy of the mass in R' is

$$K = \frac{1}{2} \iiint_{R'} \dot{x}_i' \dot{x}_i' \rho' \, dV' \ . \tag{2.7}$$

We now postulate the existence of a strain energy function as follows:

Principle: There exists a function $U = U(p)$ (the strain energy per unit volume of the unstrained state) such that

$$W = \frac{dK}{dt} + \frac{d}{dt} \iiint_{R} U \, dV \ , \tag{2.8}$$

where R is the region with boundary S in x-space corresponding to R'.

We have from (2.6), (2.5)

$$(2.9) \quad W = \iiint_{R'} \left[\frac{\partial}{\partial x_k'}(\tau_{ik}\dot{x}_i') + F_i\dot{x}_i'\right] dV'$$

$$= \iiint_{R'} \left[\left(\frac{\partial \tau_{ik}}{\partial x_k'} + F_i\right)\dot{x}_i' + \tau_{ik}\frac{\partial \dot{x}_i'}{\partial x_k}\right] dV'$$

$$= \iiint_{R'} \left[\rho'\ddot{x}_i'\dot{x}_i' + \tau_{ik}\frac{\partial \dot{x}_i'}{\partial x_r}\frac{\partial x_r}{\partial x_k'}\right] dV'$$

$$= \frac{dK}{dt} + \iiint_{R'} \tau_{ik}\dot{p}_{ir}p^{rk} dV'$$

where

$$p^{-1} = (p^{ik})$$

is the matrix reciprocal to p. On the other hand

$$(2.10) \quad \frac{d}{dt}\iiint_{R} U(p)\, dV = \iiint_{R} \frac{\partial U}{\partial p_{ir}}\dot{p}_{ir}\, dV = \iiint_{R'} \frac{1}{|p|} q_{ir}\dot{p}_{ir}\, dV' ,$$

where the q_{ir} are the functions of p defined by

$$(2.11) \quad q_{ir}(p) = \frac{\partial U(p)}{\partial p_{ir}} .$$

By comparison of (2.8), (2.9), (2.10), we find for R' shrinking into a point that

$$(2.12) \quad \tau_{ik}p^{rk}\dot{p}_{ir} = \frac{1}{|p|} q_{ir}\dot{p}_{ir} .$$

We make the assumption that a particle, whatever its local deformation p and stress at a given moment, can have any given deformation rate $\dot{p}$. Then (2.12) implies that

$$\tau_{ik}p^{rk} = \frac{1}{|p|} q_{ir} .$$

Solving these equations for p we find

$$(2.13) \quad \tau_{ik} = \frac{1}{|p|} q_{ir}p_{kr}$$

or, in matrix notation,

$$(2.14) \qquad \tau = \frac{1}{|p|} qp^* \quad .$$

Relations (2.14) take the place of the stress-strain relations of the classical linear theory, supplementing the equations of motion (2.5). We next derive the equations of motion and boundary conditions in Lagrange coordinates x.

We have from (2.10), (2.8), (2.6), (2.7)

$$\frac{d}{dt}\iiint_R U(p)dV = \iiint_R q_{ir}\dot{p}_{ir}\, dV = \iint_S q_{ir}\dot{x}_i'\xi_r\, dS - \iiint_R \frac{\partial q_{ir}}{\partial x_r}\dot{x}_i'\, dV$$

$$= \iint_S q_{ir}\dot{x}_i'\xi_r\, dS - \iiint_{R'} \frac{\partial q_{ir}}{\partial x_r}\dot{x}_i' |p|^{-1}\, dV' = W - \frac{dK}{dt}$$

$$= \iint_{S'} \tau_{ik}\xi_k'\dot{x}_i'\, dS' + \iiint_{R'} [F_i - \rho'\ddot{x}_i']\dot{x}_i'\, dV' \quad .$$

At any given moment t the $\dot{x}_i'$ are independent of the p_{ik} and q_{ik} and can be taken as arbitrary functions. Taking first for the $\dot{x}_i'$ arbitrary functions vanishing on S respectively S' we find by comparison of integrands that [1]

$$(2.15) \qquad \rho'\ddot{x}_i' = F_i + \frac{1}{|p|}\frac{\partial q_{ir}}{\partial x_r} \quad .$$

Comparing next the remaining surface integrals for $\dot{x}_i'$ arbitrary on S we obtain the relations

$$(2.16) \qquad q_{ir}\xi_r\, dS = \tau_{ik}\xi_k'\, dS' \quad .$$

Here dS' is the surface element in the strained state with unit normal ξ' that originates from a surface element dS with unit normal ξ in the unstrained state.

If U is known in its dependence on p and if the F_i are known as functions of x or x' and t, we have in (2.15) a system of second order partial differential equations for the x_i' as

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[1] These equations are given by Novozhilov [1] III (34), and Reissner [5] for the case of time-independent transformations.

functions of the x_k. Stresses and strains in the ordinary sense do not enter these equations of motions. Their place is taken by the matrices q and p, which are connected by the relations

(2.17) $$q_{ik} = \frac{\partial U(p)}{\partial p_{ik}} \quad .$$

Boundary conditions of the type of restrictions on the τ'_{ik} on S' yield by (2.16) restrictions on the q_{ik} on S.

In the special case corresponding to elastic equilibrium and in the absence of external forces the x'_i are independent of t, and the equilibrium equations take the simple form

(2.18) $$\frac{\partial q_{ik}}{\partial x_k} = 0 \quad .$$

Making use of (2.17) we obtain from (2.18) a quasi-linear system of first order equations for the p_{ik}, which has to be supplemented by the obvious "compatibility conditions" for the p_{ik}

(2.19) $$\frac{\partial p_{ik}}{\partial x_r} = \frac{\partial p_{ir}}{\partial x_k} \quad .$$

The two most common types of boundary conditions are that either the displacements or the tractions vanish on portions of S. In the first case we have $x'_i - x_i = 0$ or equivalently

(2.20) $$p_{ik} - \delta_{ik} = \sigma_i \xi_k$$

with suitable proportionality factors σ_i. In the second case $\tau_{ik}\xi'_k \, dS' = 0$ or by (2.16)

(2.21) $$q_{ik}\xi_k = 0 \quad .$$

In either case the problem of determining the x'_i or p_{ik} is described completely in Lagrange coordinates.

Chapter III

Invariants of the Strain Energy Function

We assume that the strain energy density $U(p)$ is independent of the choice of the cartesian coordinate system used. This amounts to the requirement that

$$(3.1) \qquad U(p) = U(cpc^*)$$

for any orthogonal matrix c. In addition we restrict ourselves to an isotropic medium. This has as a consequence that $U(p)$ stays unchanged if a rigid motion is superimposed on the transformation $x' = f(x)$ with Jacobian matrix p, i.e.

$$(3.2) \qquad U(p) = U(cp)$$

for orthogonal c. Altogether then

$$U(p_1) = U(p_2)$$

for two matrices p_1, p_2 that are connected by a relation

$$(3.3) \qquad p_2 = cp_1C$$

where c and C are orthogonal. The necessary and sufficient condition for two matrices p_1, p_2 to be connected by a relation (3.3) with orthogonal c, C is that the matrices $p_1^* p_1$ and $p_2^* p_2$ have the same eigenvalues. Consequently our assumptions imply that $U(p)$ is a symmetric function of the eigenvalues of $p^* p = g$.

Any symmetric function of the eigenvalues of a matrix g can be expressed in terms of any three independent symmetric functions of the eigenvalues. We choose here as basic symmetric functions the

$$(3.4) \qquad s_n = [(g - 1)^n] \qquad \text{for } n = 1,2,3 \quad ,$$

where we denote generally by $[a]$ the <u>trace</u> of the matrix a. Then

$$(3.5) \qquad U(p) = V(s_1, s_2, s_3) \quad .$$

It is easy to establish the connection between the relations given here and those given customarily,[1] in which the strain matrix is taken to be

$$\zeta = \tfrac{1}{2}(g - 1) \quad .$$

If $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of ζ and I_1, I_2, I_3 the elementary symmetric functions of those eigenvalues, we have

$$(3.5a) \qquad \begin{cases} s_1 = \sum\limits_k 2\lambda_k = 2I_1 \\ s_2 = \sum\limits_k 4\lambda_k^2 = 4(I_1^2 - 2I_2) \\ s_3 = \sum\limits_k 8\lambda_k^3 = 8(I_1^3 - 3I_1 I_2 + 3I_3) \end{cases}$$

Then $U = V(s_1, s_2, s_3)$ becomes a function of the I_k. One can write U in the form $U = \psi(\zeta)$ where $\psi(\zeta) = \psi(\zeta^*)$. Then

$$\frac{\partial\psi(\zeta)}{\partial\zeta_{ik}} = \frac{\partial\psi(\zeta)}{\partial\zeta_{ki}} \qquad \text{for } \zeta = \zeta^* \quad .$$

One obtains, using this symmetry property,

$$q_{ik} = \frac{\partial\psi}{\partial\zeta_{ks}} p_{is}$$

and from (2.13) [2]

$$\tau_{ik} = |p|^{-1} \frac{\partial\psi}{\partial\zeta_{rs}} p_{is} p_{kr} = \frac{\rho'}{\rho} \frac{\partial\psi}{\partial\zeta_{rs}} p_{is} p_{kr} \quad .$$

It is easy to find the stress strain relations corresponding to a given choice of function V. We have from (2.17), (3.5)

$$dU = q_{ik} dp_{ik} = [q^* \, dp] = \sum_{n=1}^{3} V_{s_n} \, ds_n \quad .$$

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[1] See F. D. Murnaghan, [2], Chapter III.

[2] See Murnaghan, [2], p. 56.

By (3.4)

$$ds_n = n[(g-1)^{n-1}\, dg] = n[(g-1)^{n-1}\, d(p^*p)]$$

$$= n[(g-1)^{n-1}\,(dp^*)p + (g-1)^{n-1}p^*\,(dp)]$$

$$= 2n[(g-1)^{n-1}p^*\,(dp)] \quad .$$

Hence by comparison, for arbitrary dp,

$$q^* = \sum_{n=1}^{3} 2nV_{s_n}(g-1)^{n-1}p^*$$

or

$$q = \sum_{n=1}^{3} 2nV_{s_n} p(p^*p-1)^{n-1} = \sum_{n=1}^{3} 2nV_{s_n}(pp^*-1)^{n-1}p \quad . \tag{3.6}$$

It follows from (2.14) that the stress matrix τ is given by

$$\tau = |p|^{-1} \sum_{n=1}^{3} 2nV_{s_n}(pp^*-1)^{n-1}pp^* \quad . \tag{3.7}$$

This formula makes evident that the stress matrix τ is symmetric under our assumptions.

For small deformations we have

$$p = 1 + P \quad ,$$

where P is small. Then up to terms of second order

$$pp^* \approx 1+P+P^* \;, \qquad |p| \approx 1 + [P] \;, \qquad s_1 \approx [P + P^*]$$

$$\tau \approx (1 - [P])(1 + P + P^*)(2V_{s_1} + 2V_{s_1s_1}[P+P^*] + 4V_{s_2}(P+P^*))$$

$$\approx 2V_{s_1} + 2V_{s_1}(P+P^*-[P]) + 2V_{s_1s_1}[P+P^*] + 4V_{s_2}(P+P^*) \quad ,$$

where the derivatives of $V = V(s_1,s_2,s_3)$ are taken for $p = 1$ or $s_1 = s_2 = s_3 = 0$. If we require that the stress vanishes for $p = 0$, we have

$$V_{s_1} = 0 \quad . \tag{3.8a}$$

Put for $s_1 = s_2 = s_3 = 0$

$$4V_{s_1 s_1} = \lambda \quad , \qquad 4V_{s_2} = \mu \quad ; \tag{3.8b}$$

then for small deformations

$$\tau \approx \lambda[P] + \mu(P + P^*) \quad . \tag{3.8c}$$

This is the classical linear approximation for the stress strain relations with the Lamé constants λ and μ. The constants λ and μ are necessarily positive by nature.

For later purposes it will be important to have the expansion of $U(p) = U(1+P)$ in the neighborhood of $P = 0$ up to terms of fourth order in P.

According to (3.7) the stress matrix τ is expressible in terms of the symmetric matrix

$$\gamma = pp^* \quad .$$

We have

$$\tau = |\gamma|^{-1/2} \sum_{n=1}^{3} 2nV_{s_n} (\gamma-1)^{n-1}\gamma \tag{3.8d}$$

where

$$V = V(s_1, s_2, s_3) \quad , \qquad s_n = [(\gamma-1)^n] \quad .$$

We can expand τ into a Taylor series of powers of $\gamma-1$ and find from (3.8a,b)

$$\tau = (1 - \tfrac{1}{2}[\gamma-1] + \ldots)(2V_{s_1} + 2V_{s_1 s_1}[\gamma-1] + 4V_{s_2}(\gamma-1) + \ldots)$$

$$= \tfrac{\lambda}{2}[\gamma-1] + \mu(\gamma-1) + O((\gamma-1)^2) \quad .$$

This agrees with (3.8c) since

$$\gamma-1 = P + P^* + PP^* = P + P^* + O(P^2) \quad .$$

In particular for $\gamma = 1$ and arbitrary $d\gamma$

$$\tau = 0 \quad , \qquad d\tau = \tfrac{\lambda}{2}[d\gamma] + \mu\, d\gamma \quad .$$

If $d\tau = 0$ for $\gamma = 1$ and some $d\gamma$ then

$$0 = \frac{\lambda}{2}[d\gamma] + \mu\, d\gamma$$

and hence also

$$0 = [0] = (\tfrac{3}{2}\lambda + \mu)[d\gamma] \quad .$$

Since λ and μ are positive this would imply that $[d\gamma] = 0$ and thus also $d\gamma = 0$.

We can conclude that if $\gamma-1$ is sufficiently small then $\tau \neq 0$ for $\gamma-1 \neq 0$. Otherwise there would exist a sequence of matrices γ_n of the form $\gamma_n = 1+\varepsilon_n\sigma_n$, with scalars ε_n tending to zero and matrices σ_n which are bounded and bounded away from zero such that the corresponding stress matrices τ_n vanish. For a suitable subsequence the σ_n would converge towards a non-vanishing matrix $d\gamma$ for which $d\tau = 0$, in contradiction to what has been proved.

The matrix $\gamma = pp^*$ has the same invariants as the matrix $g = p^*p$. Vanishing of $\gamma-1$ means that p is orthogonal; if the strains are small then $\gamma-1$ is small. We find then that for sufficiently small strains, i.e. for sufficiently small $g-1$, the stress matrix τ can only vanish for orthogonal p. Since by (2.14) τ and q vanish simultaneously, it follows then also that $q \neq 0$ for sufficiently small strains unless p is orthogonal.

We have from Taylor's formula

$$U = V(s_1,s_2,s_3) = \sum_{i_1,i_2,i_3} \frac{1}{i_1!\,i_2!\,i_3!} V_{i_1 i_2 i_3}\, s_1^{i_1} s_2^{i_2} s_3^{i_3}$$

where

$$V_{i_1 i_2 i_3} = \left(\frac{\partial^{i_1+i_2+i_3} V}{(\partial s_1)^{i_1} (\partial s_2)^{i_2} (\partial s_3)^{i_3}} \right)_{s_1=s_2=s_3=0} \quad .$$

We have for $i = 1,2,3$

$$s_i = [(g-1)^i] = [(P + P^* + P^*P)^i]$$
$$= \sum_{\alpha+\beta=i} \frac{i!}{\alpha!\beta!} [(P+P^*)^\alpha (P^*P)^\beta] = \sum_{\alpha+\beta=i} s_{\alpha\beta}$$

where

$$(3.9) \qquad s_{\alpha\beta} = \frac{(\alpha+\beta)!}{\alpha!\beta!} [(P + P^*)^\alpha (P^*P)^\beta] \quad .$$

Then

$$s_1^{i_1} s_2^{i_2} s_3^{i_3} = (s_{10}+s_{01})^{i_1} (s_{20}+s_{11}+s_{02})^{i_2} (s_{30}+s_{21}+s_{12}+s_{03})^{i_3}$$
$$= i_1!i_2!i_3! \sum \prod_{i+k\leq 3} \frac{s_{ik}^{\alpha_{ik}}}{\alpha_{ik}!}$$

where the sum is extended over all sets of non-negative integers $(\alpha_{10},\alpha_{01},\alpha_{20},\alpha_{11},\alpha_{02},\alpha_{30},\alpha_{21},\alpha_{12},\alpha_{03})$ with

$$(3.10) \quad \alpha_{10}+\alpha_{01} = i_1, \quad \alpha_{20}+\alpha_{11}+\alpha_{02} = i_2, \quad \alpha_{30}+\alpha_{21}+\alpha_{13}+\alpha_{03} = i_3.$$

We finally arrive then at the formula

$$(3.11) \qquad U = \sum V_{i_1 i_2 i_3} \prod_{i+k\leq 3} \frac{s_{ik}^{\alpha_{ik}}}{\alpha_{ik}!}$$

where the sum is extended over all vectors $(\alpha_{10},\ldots,\alpha_{03})$ whose nine components are non-negative integers and where i_1,i_2,i_3 are the functions of the α_{ik} defined by (3.10).

By (3.9) the expression $s_{\alpha\beta}$ is a form in the elements of the matrix P of degree $\alpha+2\beta$. Hence the general term in the sum (3.11) representing U is a form of degree

$$(3.12) \quad N = \sum_{i+k\leq 3} (i+2k)\alpha_{ik}$$
$$= \alpha_{10} + 2(\alpha_{01}+\alpha_{20}) + 3(\alpha_{11}+\alpha_{30}) + 4(\alpha_{02}+\alpha_{21}) + 5\alpha_{12} + 6\alpha_{03}.$$

Each set of non-negative integers satisfying this relation furnishes a homogeneous term of degree N in the expansion for U. Without attempting to determine in general the number of terms of degree N it is easy to find successively all solutions α_{ik} of (3.12) for N = 0,1,2,3,4. One finds in this way

$$U = V_{000} + V_{100}s_{10} + V_{100}s_{01} + \frac{1}{2} V_{200}s_{10}^2 + V_{010}s_{20}$$
$$+ \frac{1}{6} V_{300}s_{10}^3 + V_{200}s_{10}s_{01} + V_{110}s_{10}s_{20} + V_{010}s_{11} + V_{001}s_{30}$$
$$+ \frac{1}{24} V_{400}s_{10}^4 + \frac{1}{2} V_{300}s_{10}^2 s_{01} + \frac{1}{2} V_{210}s_{10}^2 s_{20} + V_{110}s_{10}s_{11}$$
$$+ V_{101}s_{10}s_{30} + V_{110}s_{01}s_{20} + \frac{1}{2} V_{200}s_{01}^2 + \frac{1}{2} V_{020}s_{20}^2$$
$$+ V_{010}s_{02} + V_{001}s_{21} + \text{(terms of higher order)} \quad .$$

Without restriction of generality we can assume that $V_{000} = 0$. Moreover by (3.8a,b)

$$V_{100} = V_{s_1} = 0 \ , \quad 4V_{200} = \lambda \ , \quad 4V_{010} = \mu \quad .$$

The cubic terms contain 3 additional constants

$$A = \frac{1}{6} V_{300} \ , \quad B = V_{110} \ , \quad C = V_{001}$$

and the fourth degree terms, 4 further constants

$$D = \frac{1}{24} V_{400} \ , \quad E = \frac{1}{2} V_{210} \ , \quad F = V_{101} \ , \quad G = \frac{1}{2} V_{020} \quad .$$

We write

$$(3.1) \qquad U = U^2(P) + U^3(P) + U^4(P) + \ldots$$

where $U^k(P)$ is a form of degree k. Here

$$U^2 = \frac{\lambda}{8} [P+P^*]^2 + \frac{\mu}{4} [(P+P^*)^2]$$

$$U^3 = \frac{\lambda}{4} [P+P^*][P^*P] + \frac{\mu}{2} [(P+P^*)P^*P] + A[P+P^*]^3$$
$$+ B[P+P^*][(P+P^*)^2] + C[(P+P^*)^3]$$

$$U^4 = \frac{\lambda}{8}[P^*P]^2 + \frac{\mu}{4}[(P^*P)^2] + 3A[P+P^*]^2[P^*P]$$
$$+ 2B[P+P^*][(P+P^*)P^*P] + B[P^*P][(P+P^*)^2] + D[P+P^*]^4$$
$$+ E[P+P^*]^2[(P+P^*)^2] + F[P+P^*][(P+P^*)^3] + G[(P+P^*)^2]^2$$
$$+ 3C[(P+P^*)^2P^*P] \quad .$$

For any matrix $\xi = (\xi_{ik})$ we use the symbol D_ξ for the differential operator defined by

(3.14) $$D_\xi = \xi_{ik}\frac{\partial}{\partial p_{ik}} \quad .$$

We have for independent matrices $\xi, \eta, \zeta, \sigma, \ldots$ and for $p = 1$

$$D_\xi U = D_\xi U^1 = U^1(\xi) \quad , \qquad D_\xi D_\eta U = D_\xi D_\eta U^2 = U^2(\xi,\eta) \quad ,$$
$$D_\xi D_\eta D_\zeta U = D_\xi D_\eta D_\zeta U^3 = U^3(\xi,\eta,\zeta) \quad ,$$
$$D_\xi D_\eta D_\zeta D_\sigma U = D_\xi D_\eta D_\zeta D_\sigma U^4 = U^4(\xi,\eta,\zeta,\sigma) \quad .$$

Here for the form $U^k(P)$ of degree k we have in $U^k(\xi^1,\xi^2,\ldots,\xi^k)$ a polar form of U^k, which has the value $k!U^k(P)$ for $\xi^1=\xi^2=\ldots=\xi^k=P$, and is symmetric in $\xi^1,\xi^2,\ldots,\xi^k$. It is simpler to write the expressions for these polar forms, which have the k-derivatives of U with respect to all the p_{ik} as coefficients, than to give expressions for the individual coefficients. We find immediately by polarisation from the expressions derived for the $U^k(P)$ that for $p = 1$

(3.15a) $$D_\xi D_\eta U = U^2(\xi,\eta) = \sum_{\xi,\eta}\left\{\frac{\lambda}{8}[\xi+\xi^*][\eta+\eta^*] + \frac{\mu}{4}[(\xi+\xi^*)(\eta+\eta^*)]\right\}$$
$$= \lambda[\xi][\eta] + \mu[\xi\eta + \xi\eta^*]$$

(3.15b) $$D_\xi D_\eta D_\zeta U = U^3(\xi,\eta,\zeta)$$
$$= \sum_{\xi,\eta,\zeta}\left\{\frac{\lambda}{4}[\xi+\xi^*][\eta^*\zeta] + \frac{\mu}{2}[(\xi+\xi^*)\eta^*\zeta]\right.$$
$$+ A[\xi+\xi^*][\eta+\eta^*][\zeta+\zeta^*] + B[\xi+\xi^*][(\eta+\eta^*)(\zeta+\zeta^*)]$$
$$\left. + C[(\xi+\xi^*)(\eta+\eta^*)(\zeta+\zeta^*)]\right\}$$

$$(3.15c)\quad D_\xi D_\eta D_\zeta D_\sigma U = U^4(\xi,\eta,\zeta,\sigma)$$

$$= \sum_{\xi,\eta,\zeta,\sigma} \Big\{ \frac{\lambda}{8}\,[\xi^*\eta][\zeta^*\sigma] + \frac{\mu}{4}\,[\xi^*\eta\zeta^*\sigma]$$

$$+ 3A[\xi+\xi^*][\eta+\eta^*][\zeta^*\sigma] + 2B[\xi+\xi^*][(\eta+\eta^*)\zeta^*\sigma]$$

$$+ B[\xi^*\eta][(\zeta+\zeta^*)(\sigma+\sigma^*)]$$

$$+ D[\xi+\xi^*][\eta+\eta^*][\zeta+\zeta^*][\sigma+\sigma^*]$$

$$+ E[\xi+\xi^*][\eta+\eta^*][(\zeta+\zeta^*)(\sigma+\sigma^*)]$$

$$+ F[\xi+\xi^*][(\eta+\eta^*)(\zeta+\zeta^*)(\sigma+\sigma^*)]$$

$$+ G[(\xi+\xi^*)(\eta+\eta^*)][(\zeta+\zeta^*)(\sigma+\sigma^*)]$$

$$+ 3C[(\xi+\xi^*)(\eta+\eta^*)\zeta^*\sigma]\Big\} .$$

Here the sums on the right are to be extended over all permutations of ξ,η respectively of ξ,η,ζ respectively of ξ,η,ζ,σ.

We notice in particular that

$$\left(\frac{\partial q_{ik}}{\partial p_{rs}}\right)_{p=1} \xi_{ik}\eta_{rs} = \left(\frac{\partial^2 U(p)}{\partial p_{ik}\,\partial p_{rs}}\right)_{p=1} \xi_{ik}\eta_{rs} = D_\xi D_\eta U$$

$$= U^2(\xi,\eta) = \lambda[\xi][\eta] + \mu[\xi\eta + \xi\eta^*]$$

$$= \lambda\xi_{ii}\eta_{kk} + \mu\xi_{ik}(\eta_{ik} + \eta_{ki}) .$$

Having the expressions for the derivatives of U at p = 1 we can easily derive expressions for the derivatives of U(p) at any orthogonal p = c. We have indeed by (3.2)

$$(3.17a)\quad \left(\frac{\partial q_{ik}(p)}{\partial p_{rs}}\right)_{p=c} \xi_{ik}\eta_{rs} = \left(\frac{\partial^2 U(p)}{\partial p_{ik}\,\partial p_{rs}}\right)_{p=c} \xi_{ik}\eta_{rs}$$

$$= (D_\xi D_\eta U(p))_{p=c}$$

$$= \left(\frac{\partial^2}{\partial\alpha\,\partial\beta}\, U(c + \alpha\xi + \beta\eta)\right)_{\alpha=\beta=0}$$

$$= \left(\frac{\partial^2}{\partial\alpha\,\partial\beta}\, U(1 + c^*\alpha\xi + c^*\beta\eta)\right)_{\alpha=\beta=0}$$

$$= \left(D_{c^*\xi}\, D_{c^*\eta}\, U(p)\right)_{p=1} = U^2(c^*\xi, c^*\eta)$$

with U^2 given by (3.15a). Similarly for c orthogonal

(3.17b) $$[D_\xi D_\eta D_\zeta U(p)]_{p=c} = U^3(c^*\xi,\ c^*\eta,\ c^*\zeta) \quad .$$

Chapter IV

The Second Variation of the Strain Energy. Uniqueness at Equilibrium for Prescribed Boundary Displacements.

We consider an equilibrium position $x' = x^{o'}(x)$ of the material maintained by certain body forces F_i^o and certain prescribed tractions or displacements on the boundary S_o' of the region R_o' filled by the material in the strained state. Let

$$(4.1) \qquad p_{ik}^o = \frac{\partial x_i^{o'}}{\partial x_k} \quad , \qquad q_{ik}^o = \left(\frac{\partial U(p)}{\partial p_{ik}}\right)_{p=p^o} \quad .$$

Then by (2.15), (2.16)

$$(4.2) \qquad \frac{\partial q_{ir}^o}{\partial x_r} + |p^o| F_i = 0 \qquad \text{in } R$$

$$(4.3) \qquad q_{ir}^o \xi_r dS = \tau_{ik}^o \xi_k^{o'} \, dS_o' \qquad \text{for} \quad x \text{ on } S, \; x^{o'} \text{ on } S_o' \quad .$$

Let $x' = x'(x)$ be an arbitrary transformation (not necessarily corresponding to an equilibrium state). We compute the difference of the total strain energies corresponding to the transformations x' and $x^{o'}$. We have, expanding $U(p)$ by Taylor's theorem,

$$(4.4) \qquad \iiint_R U(p)dV - \iiint_R U(p^o)dV = A + B \quad ,$$

where

$$A = \iiint_R \left(\frac{\partial U(p)}{\partial p_{ik}}\right)_{p=p^o} (p_{ik} - p_{ik}^o) \, dV$$

$$B = \frac{1}{2} \iiint_R \left(\frac{\partial^2 U(p)}{\partial p_{ik} \partial p_{rs}}\right)_{p=p^o+\theta(p-p^o)} (p_{ik} - p_{ik}^o)(p_{rs} - p_{rs}^o) dV$$

and $\theta = \theta(x,$ lies between 0 and 1. The quantity A represents the first variation of the total strain energy, while B (at least for infinitesimal $x'-x^{o'}$) represents the second variation.

Using the equilibrium conditions (4.2), (4.3) we have

$$(4.5)\qquad A = \iiint_R q^o_{ik} \frac{\partial(x'_i - x^{o\prime}_i)}{\partial x_k} dV$$

$$= \iiint_R |p^o| F_i (x'_i - x^{o\prime}_i) dV + \iint_S q^o_{ik} \xi_k (x'_i - x^{o\prime}_i) dS$$

Assume that the forces acting on each volume element in the strained state are prescribed, i.e. that

$$F_i \, dV' = F_i |p| dV$$

is known. Let moreover on a portion of the boundary the stresses acting on a surface element

$$\tau_{ik} \xi'_k dS' = q_{ik} \xi_k dS$$

be given, whereas on the remainder of the boundary the displacements x'_i are prescribed. We can define for any transformation $x' = x'(x)$ the potential energy by

$$P = \iiint_R U(p) dV - \iiint_R F_i |p| (x'_i - x_i) dV - \iint_{S_1} q_{ik} \xi_k (x'_i - x_i) dS$$

where p is the Jacobian matrix determined by $x'(x)$. S_1 is that portion of S for which the stresses are prescribed and $F_i |p| dV$ and $q_{ik}\xi_k dS$ have the prescribed values. If on the remaining portion of S the displacements x' and x'^o agree we have for the difference in the potential energies corresponding to the transformations $x'(x)$ and $x'^o(x)$

$$P - P^o = \iiint_R (U(p) - U(p^o)) dx - \iiint_R F_i |p| (x'_i - x'^o_i) dV - \iint_{S_1} q_{ik} \xi_k (x'_i - x'^o_i) dS$$

$$= B \quad .$$

Since B is quadratic in the first derivatives of $x' - x'^o$ we have that the potential energy is stationary for the equilibrium state.[1]

[1] See Reissner [5] for a different formulation.

The second variation B is of the form

$$B = \iiint_R Q\, dV \tag{4.6}$$

with

$$Q = \frac{1}{2}\left(\frac{\partial^2 U(p)}{\partial p_{ik}\partial p_{rs}}\right)_{p=\bar{p}} (p_{ik} - p^o_{ik})(p_{rs} - p^o_{rs}) \tag{4.7}$$

$$\bar{p} = p^o + \theta(p-p^o) \quad . \tag{4.8}$$

If B is non-negative for all displacements $x'-x^{o'}$ of a certain type, which do no work, then the total strain energy will be a minimum in the equilibrium state for all those admitted x'. This would clearly be the case if the quadratic form Q in the $p_{ik}-p^o_{ik}$ is positive definite or semi-definite. Now in the notation (3.14)

$$Q = \frac{1}{2}\,(D_\xi D_\eta U(p))_{\substack{\xi=\eta=p-p^o \\ p=\bar{p}}} \quad . \tag{4.9}$$

For the special value $\bar{p} = 1$ we have from (3.15a)

$$2Q = \lambda[\xi]^2 + \mu[\xi\xi + \xi\xi^*] = \lambda\xi_{ii}\xi_{kk} + \mu(\xi_{ik}\xi_{ik} + \xi_{ki}\xi_{ki}) \tag{4.10}$$
$$= \lambda[\xi]^2 + \frac{\mu}{2}\,[(\xi + \xi^*)(\xi^* + \xi)] \geq 0 \quad ,$$

where $\xi = p-p^o$. This form in ξ is positive semi-definite. It vanishes whenever ξ is a skew symmetric matrix. There is no reason to assume that Q will still be semi-definite when $\bar{p}$ is just close to 1 instead of being equal to 1. In spite of the expected indefinite character of Q there are cases where it is possible to establish that at least B is non-negative. This type of situation has been studied in recent years by Gårding, Aronszajn and others for general linear elliptic systems.

We observe that the form

$$\lambda\xi_{ii}\xi_{kk} + \mu(\xi_{ik}\xi_{ik} + \xi_{ik}\xi_{ki}) - \mu(\xi_{ik}\xi_{ki} - \xi_{ii}\xi_{kk}) \tag{4.11}$$
$$= (\lambda + \mu)(\xi_{ii})^2 + \mu\xi_{ik}\xi_{ik}$$

The second variation B is of the form

$$B = \iiint_R Q\, dV \tag{4.6}$$

with

$$Q = \frac{1}{2}\left(\frac{\partial^2 U(p)}{\partial p_{ik}\partial p_{rs}}\right)_{p=\overline{p}}(p_{ik}-p^o_{ik})(p_{rs}-p^o_{rs}) \tag{4.7}$$

$$\overline{p} = p^o + \theta(p-p^o) \quad . \tag{4.8}$$

If B is non-negative for all displacements $x'-x^{o'}$ of a certain type, which do no work, then the total strain energy will be a minimum in the equilibrium state for all those admitted x'. This would clearly be the case if the quadratic form Q in the $p_{ik}-p^o_{ik}$ is positive definite or semi-definite. Now in the notation (3.14)

$$Q = \frac{1}{2}\left(D_\xi D_\eta U(p)\right)_{\substack{\xi=\eta=p-p^o \\ p=\overline{p}}} \quad . \tag{4.9}$$

For the special value $\overline{p} = 1$ we have from (3.15a)

$$\begin{aligned} 2Q &= \lambda[\xi]^2 + \mu[\xi\xi + \xi\xi^*] = \lambda\xi_{ii}\xi_{kk} + \mu(\xi_{ik}\xi_{ik} + \xi_{ki}\xi_{ki}) \\ &= \lambda[\xi]^2 + \frac{\mu}{2}\,[(\xi + \xi^*)(\xi^* + \xi)] \geq 0 \quad , \end{aligned} \tag{4.10}$$

where $\xi = p-p^o$. This form in ξ is positive semi-definite. It vanishes whenever ξ is a skew symmetric matrix. There is no reason to assume that Q will still be semi-definite when $\overline{p}$ is just close to 1 instead of being equal to 1. In spite of the expected indefinite character of Q there are cases where it is possible to establish that at least B is non-negative. This type of situation has been studied in recent years by Gårding, Aronszajn and others for general linear elliptic systems.

We observe that the form

$$\begin{aligned} \lambda\xi_{ii}\xi_{kk} + \mu(\xi_{ik}\xi_{ik} + \xi_{ik}\xi_{ki}) - \mu(\xi_{ik}\xi_{ki} - \xi_{ii}\xi_{kk}) \\ = (\lambda+\mu)(\xi_{ii})^2 + \mu\xi_{ik}\xi_{ik} \end{aligned} \tag{4.11}$$

is positive definite in ξ, since λ and μ are positive constants. The same holds then for all quadratic forms with neighboring coefficients. Assuming $U = U(p)$ to have continuous second derivatives with respect to the p_{ik} there exists then an ε only depending on the choice of strain energy function U such that

$$\frac{1}{2} D_\xi D_\xi U(\bar{p}) - \frac{\mu}{2}([\xi^2] - [\xi]^2)$$

$$= \frac{1}{2} D_\xi D_\xi U(\bar{p}) - \frac{\mu}{2}(\xi_{ik}\xi_{ki} - \xi_{ii}\xi_{kk}) > 0$$

for

$$\xi \neq 0, \quad (\bar{p}^* - 1)(\bar{p} - 1) < \varepsilon .$$

Hence

(4.12) $$Q > \frac{\mu}{2}\left((p_{ik} - p^o_{ik})(p_{ki} - p^o_{ki}) - (p_{ii} - p^o_{ii})(p_{kk} - p^o_{kk})\right)$$

$$= \frac{\mu}{2} \frac{\partial}{\partial x_i}\left((p_{ik} - p^o_{ik})(x'_k - x^{o\prime}_k) - (p_{kk} - p^o_{kk})(x'_i - x^{o\prime}_i)\right)$$

for $p \neq p^o$ and

(4.13) $$[(p^* - 1)(p - 1)] < \varepsilon , \quad [(p^{o*} - 1)(p^o - 1)] < \varepsilon .$$

It follows that under the assumptions (4.13)

$$B > \frac{\mu}{2} \iint_S \left((p_{ik} - p^o_{ik})(x'_k - x^{o\prime}_k) - (p_{kk} - p^o_{kk})(x'_i - x^{o\prime}_i)\right) \xi_i \, dS$$

unless

$$p_{ik} = p^o_{ik} \quad \text{in} \quad R .$$

If the displacements x' and $x^{o\prime}$ are equal on the boundary and if p and p^o lie in the ε-neighborhood of the identity described by (4.13), we find that $B > 0$ unless $p = p^o$ in R. The identity of p and p^o implies that the $x'_i - x^{o\prime}_i$ are constant and hence vanish, since they vanish on the boundary S. This yields the theorem:

In the absence of external volume forces any change from the equilibrium state that leaves the boundary points fixed leads to an increase in total strain energy, as long as only states in an ε-neighborhood of the identity are considered. Here the ε-neighborhood of a transformation with Jacobian matrix p^o is defined by

$$[(p^* - p^{o*})(p - p^o)] < \varepsilon.$$

Since then for given boundary displacements any state differing from a given equilibrium state (in an ε-neighborhood of the identity) corresponds to a larger total strain energy, we arrive at a uniqueness theorem:

For given displacements at the boundary there can exist at most one equilibrium state in the ε-neighborhood of the identity. Here ε depends exclusively on the choice of strain energy function.

It would be desirable to have a uniqueness theorem applying to transformations x' not restricted to the ε-neighborhood of the identity. It is clear that the validity of such a theorem would depend on the behavior of $U(p)$ for matrices p removed from the identity.

Chapter $\underline{V}$

Two Dimensional Bending of a Thin Plate

In this chapter we only consider <u>plane</u> deformations of the form

(5.1) $$x_1' = x_1'(x_1,x_2)\ ,\quad x_2' = x_2'(x_1,x_2),\qquad x_3' = x_3\ .$$

Then the Jacobian matrix p is of the form

(5.2) $$p = \begin{pmatrix} p_{11} & p_{12} & 0 \\ p_{21} & p_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows for the matrix $\gamma = pp^*$ that

$$\gamma_{13} = \gamma_{23} = \gamma_{31} = \gamma_{32} = 0,\ \gamma_{33} = 1.$$

Since by (3.8) the matrix τ is a function of γ we have then also

$$\tau_{13} = \tau_{23} = \tau_{31} = \tau_{32} = 0,\ \tau_{33} = 2|p|^{-1}V_{s_1}(s_1,s_2,s_3).$$

Thus for $q = |p|\tau(p^*)^{-1}$

(5.3) $$q_{13} = q_{23} = q_{31} = q_{32} = 0,\ q_{33} = 2V_{s_1}(s_1,s_2,s_3).$$

For a matrix p of the special form (5.2) the s_j are determined completely by $p_{11}, p_{12}, p_{21}, p_{22}$. Once those quantities are known we can find all of the τ_{ik} and q_{ik} explicitly.

Using (5.3) and the fact that all p_{ik}, q_{ik} are functions of x_1, x_2 only, the equilibrium equations (2.18) in the absence of volume forces reduce to the two conditions

$$(5.4) \qquad \frac{\partial q_{i1}}{\partial x_1} + \frac{\partial q_{i2}}{\partial x_2} = 0 \quad \text{for } i = 1,2$$

In addition we have the compatibility conditions

$$(5.5) \qquad \frac{\partial p_{i1}}{\partial x_2} - \frac{\partial p_{i2}}{\partial x_1} = 0 \quad \text{for } i = 1,2.$$

The matrices p and q are connected by the relations

$$(5.6) \qquad q_{ik} = \frac{\partial U(p)}{\partial p_{ik}}$$

In forming the q_{ik} with $i,k = 1,2$ the matrix p can already be assumed to be of the special form (5.2).

We consider now the case of a medium which in the unstrained state fills the region with rectangular cross section

$$(5.7) \qquad 0 < x_1 < a \ , \ -h < x_2 < +h \ .$$

The boundary requirements on the "horizontal" sides $x_2 = \pm h$ shall be the absence of tractions, or by (2.21)

$$(5.8) \qquad q_{12} = q_{22} = 0 \quad \text{for } x_2 = \pm h \ .$$

Boundary conditions on the vertical sides $x_1 = 0, a$ would have to be imposed as well but will not be specified at present.

One trivial family of solutions is obtained by taking for x_1', x_2' linear functions of x_1, x_2. The matrices p and q will then have constant elements, so that (5.4),(5.5) are satisfied automatically. The constant matrix p has only to be chosen in accordance with the boundary condition (5.8) that $q_{12}(p) = q_{22}(p) = 0$.

We are interested in other types of solutions corresponding to a buckled state of the medium for small thickness h.

We assume that there exists a solution of (5.4),(5.5),(5.6), (5.8) which depends analytically on x_2 and h. The matrix p can then be expanded into a power series

$$(5.9) \qquad p = \sum_{\alpha,\beta=0}^{\infty} \frac{1}{\alpha!\beta!} p^{\alpha\beta} x_2^{\alpha} h^{\beta}$$

where

$$p^{\alpha\beta} = p^{\alpha\beta}(x_1) = \left(\frac{\partial^{\alpha+\beta} p(x_1,x_2,h)}{\partial x_2{}^{\alpha} \partial h^{\beta}} \right)_{x_2=h=0.}$$

Because p is of the form (5.2)

$$(5.10a) \quad p_{i3}^{\alpha\beta} = p_{3k}^{\alpha\beta} = 0 \text{ for } \alpha + \beta > 0;\ i,k = 1,2,3.$$

Similarly

$$(5.11) \qquad q = q(x_1,x_2,h) = \sum_{\alpha,\beta=0}^{\infty} \frac{1}{\alpha!\beta!} q^{\alpha\beta} x_2^{\alpha} h^{\beta}$$

where the matrices $q^{\alpha\beta}$ depend on x_1 alone. Here

$$(511a) \qquad q_{i3}^{\alpha\beta} = q_{3k}^{\alpha\beta} = 0 \text{ for } i,k = 1,2.$$

The differential equations (5.4),(5.5) become

$$(5.12) \qquad \frac{d\, q_{i1}^{\alpha\beta}}{dx_1} + q_{i2}^{\alpha+1\ \beta} = 0,\ \frac{d\, p_{i2}^{\alpha\beta}}{dx_1} - p_{i1}^{\alpha+1\ \beta} = 0 \quad \text{for } i = 1,2.$$

The boundary conditions (5.8) yield the additional equations

$$(5.12a) \qquad \sum_{\alpha+\beta=\gamma} \frac{q_{i2}^{\alpha\beta}}{\alpha!\beta!} = \sum_{\alpha+\beta=\gamma} (-1)^{\alpha} \frac{q_{i2}^{\alpha\beta}}{\alpha!\beta!} = 0$$

$$\text{for } i = 1,2, \text{ and } \gamma = 0,1,2,\ldots$$

Adding and subtracting these two relations and expressing $q_{i2}^{\alpha+1\ \beta}$ in terms of $q_{i1}^{\alpha\beta}$ by (5.12) we arrive at the two equivalent conditions

$$(5.13a) \qquad \binom{\gamma}{0} q_{i1}^{o\ \gamma} + \binom{\gamma}{2} q_{i2}^{2\ \gamma-2} + \binom{\gamma}{4} q_{i2}^{4\ \gamma-4} + \ldots = 0$$

$$(5.13b) \qquad \binom{\gamma+1}{1} q_{i1}^{o\ \gamma} + \binom{\gamma+1}{3} q_{i1}^{2\ \gamma-2} + \binom{\gamma+1}{5} q_{i1}^{4\ \gamma-4} + \ldots = \text{const.} = C_i^{\gamma}$$

for $i = 1,2$ and $\gamma = 0,1,2,\ldots$

Equations (5.12),(5.13a,b) are supplemented by the recursion formulae obtained from (5.6) that connect the $q^{\alpha\beta}$ with the $p^{\alpha\beta}$. We have first of all

(5.14a) $$q_{ik}^{oo} = \left(\frac{\partial U}{\partial p_{ik}}\right)_{p = p^{oo}}$$

(5.14b) $$q^{lo} = \left(\frac{\partial q}{\partial x_2}\right)_{x_2 = h = o}$$

$$= \left(\frac{\partial q}{\partial p_{rs}} \frac{\partial p_{rs}}{\partial x_2}\right)_{x_2=h=o} = \left(\frac{\partial q}{\partial p_{rs}}\right)_{p=p^{oo}} p_{rs}^{lo}$$

$$= (D_{p^{lo}}\, q)_{p=p^{oo}} \quad .$$

Similarly

(5.14c) $$q^{ol} = (D_{p^{ol}}\, q)_{p=p^{oo}}$$

(5.14d) $$q^{2o} = (D_{p^{lo}} D_{p^{lo}}\, q + D_{p^{2o}}\, q)_{p=p^{oo}}$$

(5.14e) $$q^{ll} = (D_{p^{lo}} D_{p^{ol}}\, q + D_{p^{ll}}\, q)_{p=p^{oo}}$$

(5.14f) $$q^{o2} = (D_{p^{ol}} D_{p^{ol}}\, q + D_{p^{o2}}\, q)_{p=p^{oo}}$$

etc.

For the quantities of lowest order we have from (5.13,a,b) for $\gamma = 0$

(5.15) $$q_{12}^{oo} = 0, \quad q_{il}^{oo} = \text{const.} = C_i^o \text{ for } i = 1,2$$

The p^{oo} would then have to be determined from relations (5.14a). For known p_{ik}^{oo}, q_{ik}^{oo} the values of the q_{12}^{lo} and p_{il}^{lo} follow from (5.12) for $\alpha=\beta=0$. The remaining components p_{12}^{lo} would follow from the relations (5.14b) from the known q_{12}^{lo}. From (5.13a,b) for $\gamma = 1$ we have

(5.16) $$q_{12}^{ol} = 0, \quad q_{il}^{ol} = \text{const.} = \tfrac{1}{2}\, C_i^l \quad \text{for } i = 1,2.$$

For known p^{oo}, q^{ol} we have in (5.14c) a system of linear equations for the p^{ol}_{ik}. If this system has a unique solution we can proceed in the same manner and get successively all $p^{\alpha\beta}$, $q^{\alpha\beta}$ if p^{oo} has been determined and the constants C_i^γ are known.

The values of the constants C_i^γ are determined if suitable boundary conditions are prescribed on the vertical sides $x_1=0,a$. These constants are indeed determined uniquely by the resultant stress force applied to the vertical faces. We have from (5.13b)

$$(5.17)\qquad \int_{-h}^{+h} q_{i1}(x_1,x_2)dx_2 = \int_{-h}^{+h} \sum_{\alpha,\beta=o}^{\infty} \frac{1}{\alpha!\beta!} q_{i1}^{\alpha\beta}(x_1)x_2^{\alpha}h^{\beta}\, dx_2$$

$$= \sum_{\alpha,\beta=o}^{\infty} \frac{1}{(\alpha+1)!\beta!} q_{i1}^{\alpha\beta}(x_1)(1+(-1)^{\alpha})h^{\alpha+\beta+1}$$

$$= 2\sum_{\gamma=o}^{\infty} \frac{1}{(\gamma+1)!} C_i^{\gamma}h^{\gamma+1}$$

For $x_1 = a$ we have by (2.16) for the components of the resultant of the stresses acting on the face $x_1 = a$ (per unit of length of x_3)

$$(5.18)\qquad \int \tau_{ik}\xi'_k\, ds' = \int_{-h}^{h} q_{i1}(a,x_2)dx_2$$

If this resultant stress is given for each h the constants C_i^γ are determined.

The procedure outlined determines for known p^{oo} and C_i^γ uniquely all $p^{\alpha\beta}$, $q^{\alpha\beta}$ and hence also p and q, <u>provided</u> equation (5.14c) can be solved for p^{ol}, equation (5.14f) for p^{o2}, etc. This is the case as long as

$$(D_\xi\, q)_{p=p^{oo}} = \left(\frac{\partial q}{\partial p_{rs}}\right)_{p=p^{oo}} \xi_{rs}$$

cannot vanish for $\xi \neq 0$. The interesting cases corresponding to buckling occur, when this condition is violated. These are the cases where the determinant of the

$$\frac{\partial q_{ik}}{\partial p_{ik}} = \frac{\partial^2 U}{\partial p_{ik}\partial p_{rs}}$$

vanishes for $p = p^{\infty}$.

This exceptional situation always arises, when

(5.19) $\quad q^{oo} = 0,$

i.e. when the stresses vanish to lowest order in h. Condition (5.19) has been shown in Chapter 3 to be equivalent to orthogonality of the matrix p^{oo}.

Let now

(5.20) $\quad p^{oo} = c \qquad \text{where } cc^* = 1.$

By (3.17a),(3.15a)

(5.21) $$\left(\frac{\partial q_{ik}}{\partial p_{rs}}\right)_{p=c} \xi_{ik}\eta_{rs} = [(D_\xi q)_{p=c}\eta^*] = (D_\xi D_\eta U(p))_{p=c}$$

$$= \left(\frac{\partial q_{rs}}{\partial p_{ik}}\right)_{p=c} \xi_{ik}\eta_{rs}$$

$$= U^2(c^*\xi, c^*\eta) = \lambda[c^*\xi][c^*\eta]+\mu[c^*\xi c^*\eta + \xi^*\eta].$$

The relation

(5.22) $\quad (D_\xi q)_{p=c} = 0$

for a certain ξ is equivalent with

$$U^2(c^*\xi, c^*\eta) = 0 \quad \text{for all } \eta.$$

For any matrix a the expression

$$[a\eta] = a_{ki}\eta_{ik}$$

vanishes identically in η if and only if a = 0. Hence (5.22) is equivalent to

$$\lambda[c^*\xi]c^* + \mu(c^*\xi c^* + \xi^*) = 0$$

or to

(5.23) $\quad \lambda[c^*\xi] + \mu(c^*\xi + \xi^* c) = 0.$

Taking the trace of relation (5.23) we find

$$(3\lambda + 2\mu)[c^*\xi] = 0$$

and hence (for positive λ, μ)

$$[c^*\xi] = 0$$

and then

$$(c^*\xi) + (c^*\xi)^* = 0.$$

Thus (5.23) is equivalent to skew symmetry of $c^*\xi$, i.e. to the existence of a matrix Ω such that

(5.24) $\quad \xi = c\,\Omega, \quad \Omega^* = -\Omega$

Whenever (5.24) holds then for any

(5.24a) $\quad [(D_\xi q)_{p=c}\eta^*] = 0.$

Assume then that

$$q^{oo} = 0$$

which is consistent with (5.15). The matrix p^{oo} is then orthogonal. Since p^{oo} is of the form (5.2) also we must have an angle $\theta = \theta(x_1)$ such that

(5.26) $$p^{oo} = c = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We make the further important assumption that $\theta(x_1)$ is not constant, thus excluding trivial solutions.

Along the line x_2 = const. = 0 we have $\tan\theta = \dfrac{dx_2'}{dx_1'}$ so that θ gives the inclination of the "middle surface" in the strained state in the limit for $h = 0$. It is our object to derive a differential equation for θ.

The q_{ik}^{ol} for i,k=1,2 have constant values given by (5.16). By (5.14c),(5.21)

(5.27) $$q_{ik}^{ol}\,\xi_{ik} = \left(\frac{\partial q_{ik}}{\partial p_{rs}}\right)_{p=c} \xi_{ik} p_{rs}^{ol} = U^2(c^* p^{ol}, c^*\xi) = 0$$

when ξ is of the form (5.24). Take for Ω the matrix

(5.27a) $$\Omega = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that

(5.28) $$\xi = c\,\Omega = \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{dc}{d\theta}$$

Then by (5.27),(5.16)

$$0 = q^{o1}_{ik}\,\xi_{ik} = \tfrac{1}{2}\,c^1_i\,\xi_{i1} = \tfrac{1}{2}\,(-c^1_1 \sin\theta + c^1_2 \cos\theta)$$

for all x_1. Since θ is not constant it follows that

$$c^1_1 = c^1_2 = 0$$

and hence (see (5.11a))

(5.30) $\quad q^{o1}_{ik} = 0$ except possibly for $i=k=3$.

There is no need to determine p^{o1} completely. We have from (5.30),(5.10a),(5.14c),(5.21)

$$0 = q^{o1}_{ik}\,p^{o1}_{ik} = (D_{p^{o1}}D_{p^{o1}}U(p))_{p=c} = U^2(c^*p^{o1}, c^*p^{o1}).$$

By (4.10) the form in ξ

$$U^2(\xi,\xi) = \lambda[\xi]^2 + \tfrac{\mu}{2}[(\xi + \xi^*)(\xi^* + \xi)]$$

is semi-definite and vanishes only for skew symmetric ξ. Hence

(5.31) $\quad (c^*p^{o1}) + (c^*p^{o1})^* = 0.$

For q^{o2} we have from (5.14f) and any ξ

$$[q^{o2}\xi^*] = [(D_{p^{o1}}D_{p^{o1}}q + D_{p^{o2}}q)_{p=c}\,\xi^*].$$

For ξ given by (5.28) it follows from (5.24a) that

$$[(D_{p^{o2}}q)_{p=c}\,\xi^*] = [(D_\xi q)_{p=c}\,p^{o2^*}] = 0.$$

Consequently, using (3.17b),

(5.32) $$q^{o2}_{ik}\,\xi_{ik} = [q^{o2}\xi^*] = [(D_{p^{o1}}D_{p^{o1}}q)_{p=c}\,\xi^*]$$

$$= (D_{p^{o1}}D_{p^{o1}}D_\xi U(p))_{p=c} = U^3(c^*p^{o1}, c^*p^{o1}, c^*\xi)$$

$$= U^3(c^*p^{o1}, c^*p^{o1}, \Omega).$$

The general expression for U^3 has been computed in (3.15b). Since here all three arguments c^+p^{o1}, c^+p^{o1}, Ω are skew symmetric and each term in U^3 contains at least one factor depending on the symmetric part of one of the arguments, we clearly have

$$(5.33)\qquad q^{o2}_{ik}\,\xi_{ik} = U^3(c^*p^{o1}, c^*p^{o1}, \Omega) = 0.$$

By (5.13a,b) with $\gamma=2$

$$(533a)\qquad q^{o2}_{i2} = -q^{2o}_{i2}\ ,\ q^{o2}_{i1} = -\frac{1}{3}(q^{2o}_{i1} - c^2_i)\quad \text{for } i=1,2$$

Hence (5.33) can be given the form

$$(q^{2o}_{12}\,\xi_{12} + q^{2o}_{22}\,\xi_{22}) + \frac{1}{3}(q^{2o}_{11}\,\xi_{11} + q^{2o}_{21}\,\xi_{21})$$
$$= \frac{1}{3}(c^2_1\,\xi_{11} + c^2_2\,\xi_{21}).$$

We can write this relation in the form

$$(5.34)\qquad c^2_i\,\xi_{i1} = 2q^{2o}_{i2}\,\xi_{i2} + q^{2o}_{ik}\,\xi_{ik}\ .$$

Here by (5.12) for $\alpha=1, \beta=0$

$$q^{2o}_{i2}\,\xi_{i2} = -\frac{d\,q^{1o}_{i1}}{dx_1}\,\xi_{i2}$$

whereas by (5.14d) (in analogy to (5.32))

$$q^{2o}_{ik}\,\xi_{ik} = (D_{p^{1o}}D_{p^{1o}}D_{\xi}U(p))_{p=c}$$
$$= U^3(c^*p^{1o}, c^*p^{1o}, \Omega).$$

Hence our relation (5.34) goes over into a relation only involving the matrices p^{1o}, q^{1o}:

$$(5.35)\qquad c^2_i\,\xi_{i1} = -2\,\frac{dq^{1o}_{i1}}{dx_1}\,\xi_{i2} + U^3(c^*p^{1o}, c^*p^{1o}, \Omega).$$

In the expression (3.15b) for U^3 the terms containing the higher elastic constants A,B,C make no contribution here, since Ω is skew symmetric. Hence

$$U^3(c^*p^{1o}, c^*p^{1o}, \Omega) = 2\lambda[c^*p^{1o}][\Omega^*c^*p^{1o}]$$
$$+ 2\mu[c^*p^{1o^*}c)\Omega^*c^*p^{1o}]$$
$$= 2\lambda[c^*p^{1o}][\xi^*p^{1o}] + 2\mu\,[c^*p^{1o} + p^{1o^*}c)\xi^*p^{1o}]$$
$$= 2U^2(c^*p^{1o}, \xi^*p^{1o})$$
$$= 2(D_{p^{1o}}\,D_{c\xi^*p^{1o}}\,U(p))_{p=c} \quad = 2[q^{1o}p^{1o^*}\xi c^*]$$

For the matrices $\xi c, \Omega$ given by (5.28),(5.26), (5.27a) we have

$$\xi c^* = \Omega$$

Hence

$$U^3(c^* p^{lo}, c^* p^{lo}, \Omega) = 2[q^{lo} p^{lo^*} \Omega]$$

$$= 2(q^{lo}_{1k}\, p^{lo}_{2k} - q^{lo}_{2k}\, p^{lo}_{1k})$$

The differential equations (5.12) yield for $\alpha=\beta=0$

$$q^{lo}_{i2} = -\frac{dq^{oo}_{i1}}{dx_1} = 0, \quad p^{lo}_{i1} = \frac{dp^{oo}_{i2}}{dx_1} = \frac{dc_{i2}}{dx_1}$$

Hence

$$U^3(c^* p^{lo}, c^* p^{lo}, \Omega) = 2(q^{lo}_{11} \frac{dc_{22}}{dx_1} - q^{lo}_{21} \frac{dc_{12}}{dx_1})$$

$$= 2(q^{lo}_{11} \frac{dc_{11}}{dx_1} + q^{ol}_{21} \frac{dc_{21}}{dx_1}) = 2\, q^{lo}_{i1} \frac{dc_{i1}}{dx_1} \quad .$$

On the other hand

$$-2 \frac{dq^{lo}_{i1}}{dx_1} \xi_{i2} = 2 \frac{dq^{lo}_{i1}}{dx_1} c_{i1} \ .$$

Consequently relation (5.35) takes the form

$$(5.36) \qquad c_i^2\, \xi_{i1} = 2 \frac{d}{dx_1} (q^{lo}_{i1}\, c_{i1})$$

It remains to evaluate the term on the right hand side of (5.36) in terms of expressions involving θ. We have for any matrix η

$$[q^{lo}\eta^*] = U^2(c^* p^{lo}, c^*\eta)$$

$$= \lambda[c^* p^{lo}][c^*\eta] + \mu\, [(c^* p^{lo} + p^{lo^*} c) c^*\eta].$$

Since η is arbitrary this implies that

$$q^{lo} = \lambda[c^* p^{lo}]\, c + \mu\, (p^{lo} + c p^{lo^*} c).$$

Multiplying this relation from the left with c^* we obtain

$$c^* q^{lo} = \lambda[c^* p^{lo}] + \mu(c^* p^{lo} + p^{lo^*} c)$$

or component wise

$$c_{ri}\, q^{lo}_{rk} = \lambda[c^* p^{lo}]\, \delta_{ik} + \mu(c_{ri}\, p^{lo}_{rk} + p^{lo}_{ri}\, c_{rk}).$$

For i=k=1 and i=k=2 we obtain respectively

$$c_{r1}\, q^{10}_{r1} = \lambda[c^{*}p^{10}] + 2\mu\, c_{r1}\, p^{10}_{r1}$$

$$\begin{aligned} 0 &= \lambda[c^{*}p^{10}] + 2\mu c_{r2}\, p^{10}_{r2} \\ &= \lambda[c^{*}p^{10}] + 2\mu c_{rs}\, p^{10}_{rs} - 2\mu c_{r1}\, p^{10}_{r1} - 2\mu c_{r3}\, p^{10}_{r3} \\ &= (\lambda+2\mu)[c^{*}p^{10}] - 2\mu c_{r1}\, p^{10}_{r1}\,. \end{aligned}$$

Eliminating $[c^{*}p^{10}]$ between the two equations we find

$$\begin{aligned} c_{r1}\, q^{10}_{r1} &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\, c_{r1}\, p^{10}_{r1} \\ &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\, c_{r1}\, \frac{d\, p^{00}_{r2}}{dx_1} \\ &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\, c_{r1}\, \xi_{r2}\, \frac{d\theta}{dx_1} = -\frac{4\mu(\mu+\lambda)}{\lambda+2\mu}\, \frac{d\theta}{dx_1}\,. \end{aligned}$$

Substituting this expression into (5.28) we obtain for $\theta = \theta(x_1)$ the differential equation

$$\frac{8\mu(\mu+\lambda)}{\lambda+2\mu}\, \frac{d^2\theta}{dx_1^{\,2}} - C_1^2 \sin\theta + C_2^2 \cos\theta = 0. \tag{5.37}$$

The coefficient

$$D = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\, h^3 \tag{5.38}$$

is the "flexuräl rigidity" of the plate. (See Love [1], p.464). The constants C_i^2 were defined by (5.33a)

$$C_i^2 = q^{20}_{i1} + 3q^{02}_{i1} \qquad \text{for } i=1,2,$$

The terminal force (per unit of length of x_3) acting on the face $x_1 = a$ is by (5.18), (5.17)

$$\int_{x_1=a} \tau_{ik}\, \xi'_k\, ds' = 2 \sum_{\gamma=0}^{\infty} \frac{1}{(\gamma+1)!}\, C_i^{\gamma}\, h^{\gamma+1}$$

This is also the total force acting on any cross section $x_1 =$ const. in the strained state. Since in the case considered

Here the C_i^1 and C_i^o vanish, we have for the components of the terminal force

$$\frac{1}{3} C_i^2 h^3 + O(h^4), \quad (i = 1,2).$$

The component of this force vertical to the middle surface in the strained state (the Shear force in a cross section x_1 = const) is then given, up to terms of higher order in h, by

$$T = \frac{1}{3}(-C_1^2 \sin \theta + C_2^2 \cos \theta) h^3 + O(h^4).$$

Hence (5.37) yields up to terms of higher order.

$$\frac{d^2\theta}{dx_1{}^2} = - \frac{T}{D}$$

Since p^{oo} is orthogonal, we have also up to terms of higher order in h for $x_2 = 0$

$$ds'^2 = dx_1'^2 + dx_2'^2 = (p_{11}{}^2 + p_{21}{}^2) dx_1{}^2$$

$$= dx_1{}^2(1 + O(h))$$

Hence up to terms of higher order

$$D \frac{d^2\theta}{ds'^2} + T = 0.$$

By a suitable rigid motion we can always bring about that the terminal force acting on the end $x_1 = a$ is parallel to the x_1-axis and has components $(-R,0)$, where R is positive in case of compression. Then

$$\frac{1}{3} C_1^2 h^3 = -R, \; C_2^2 = 0$$

so that the differential equation for θ takes the familiar form

$$D \frac{d^2\theta}{dx^2} + R \sin \theta = 0.$$

Chapter VI

Buckling of thin plates.

We consider a solid, which in the unstrained state is restricted to a portion of the parallel slab

$$- h < x_3 < + h.$$

The faces in the strained state corresponding to $x_3 = \pm h$ shall be free of tractions, i.e. we assume the boundary conditions

$$q_{i3} = 0 \quad \text{for } x_3 = \pm h. \tag{6.1}$$

Let the solid be in equilibrium under the influence of certain lateral forces with external volume forces absent. To derive differential equation for the "middle surface" for $h \to 0$, we proceed exactly as in the preceding chapter. We assume that there is a family of solutions depending analytically on x_3 and h, for which p and q are given by power series expansions

$$p = \sum_{\alpha,\beta=0}^{\infty} \frac{x_3^{\alpha} h^{\beta}}{\alpha!\beta!} p^{\alpha\beta}(x_1,x_2) \tag{6.2}$$

$$q = \sum_{\alpha,\beta=0}^{\infty} \frac{x_3^{\alpha} h^{\beta}}{\alpha!\beta!} q^{\alpha\beta}(x_1,x_2) \tag{6.3}$$

The differential equations of equilibrium (2.18) become

$$q_{i1,1}^{\alpha\beta} + q_{i2,2}^{\alpha\beta} + q_{i3}^{\alpha+1\ \beta} = 0 \quad \text{for } i=1,2,3 \tag{6.4}$$

$$p_{i2,1}^{\alpha\beta} = p_{i1,2}^{\alpha\beta}\ , \quad p_{i3,1}^{\alpha\beta} = p_{i1}^{\alpha+1\ \beta}\ , \quad p_{i3,2}^{\alpha\beta} = p_{i2}^{\alpha+1\ \beta} \tag{6.5}$$

for i=1,2,3. In addition we have from the boundary conditions (6.1) for $\gamma = 0,1,2,\dots$

$$\sum_{\alpha+\beta=\gamma} \frac{q_{i3}^{\alpha\beta}}{\alpha!\beta!} = 0\ , \quad \sum_{\alpha+\beta=\gamma} (-1)^{\alpha} \frac{q_{i3}^{\alpha\beta}}{\alpha!\beta!} = 0$$

in analogy to (5.12a). The result of adding these two relations can be written

(6.6) $\quad \binom{\gamma}{0} q^{o}_{i3}{}^{\gamma} + \binom{\gamma}{2} q^{2}_{i3}{}^{\gamma-2} + \binom{\gamma}{4} q^{4}_{i3}{}^{\gamma-4} + \ldots 0.$

If we subtract the two relations and use (6.4) we obtain

$$\binom{\gamma+1}{1}(q^{o\ \gamma}_{i1,1} + q^{o\ \gamma}_{i2,2}) + \binom{\gamma+1}{3}(q^{2\ \gamma-2}_{i1,1} + q^{2\ \gamma-2}_{i2,2}) + \ldots = 0.$$

To lowest order we have from (6.6),(6.5) the nine equations

(6.8) $\quad q^{oo}_{i3} = 0, \ q^{oo}_{i1,1} + q^{oo}_{i2,2} = 0, \ p^{oo}_{i2,1} - p^{oo}_{i1,2} = 0$

for $i=1,2,3$

where the matrices q^{oo} and p^{oo} are connected by

(6.9) $\quad q^{oo} = q(p^{oo}).$

We choose again among the solutions of the system (6.8), (6.9) the ones corresponding to vanishing stresses

(6.10) $\quad q^{oo} = 0.$

Then, as we know, p^{oo} must be an orthogonal matrix. We can easily find the most general $p^{oo} = p^{oo}(x_1,x_2)$ which is orthogonal and also satisfies the last group of relations in (6.8): The equations

$$p^{oo}_{i2,1} - p^{oo}_{i1,2} = 0, \ i=1,2,3$$

are equivalent to the existence of three functions

$w_i(x_1,x_2) = x'^{oo}_i$ such that

$$p^{oo}_{ik} = w_{i,k} \quad \text{for } i=1,2,3 \text{ and } k=1,2.$$

Orthogonality of p^{oo} implies that

$$p^{oo}_{ji}\, p^{oo}_{jk} = \delta_{ik} \quad \text{for } i,k=1,2,3.$$

For $i,k = 1,2$ this yields

$$w_{j,k}\, p^{oo}_{j3} = 0 \quad \text{for } k=1,2$$

This means that the vectors $(w_1,w_2,w_3) = (x'^{oo}_1, x'^{oo}_2, x'^{oo}_3)$ for varying x_1,x_2 describe a developable surface referred to the isometric parameters x_1,x_2. Since, moreover

$$w_{j,k}\, p^{oo}_{j3} = 0 \quad \text{for } k=1,2$$

we see that the quantities p^{oo}_{i3} are the components of the unit vector normal to this developable surface.

We shall only consider here the case where this limiting middle surface is the plane $x'_3 = 0$, and where in fact the whole

transformation is the identity to lowest order, i.e. we take

(6.11) $\quad p^{oo}_{ik} = \delta_{ik}$

This is consistent with all relations (6.8).

We proceed to the determination of p^{ol} and q^{ol}.

We have from (6.5),(6.6),(6.7)

(6.12) $\quad q^{ol}_{i3} = 0, \; q^{ol}_{i1,1} + q^{ol}_{i2,2} = 0, \; p^{ol}_{i2,1} - p^{ol}_{i1,2} = 0$

for i=1,2,3.

Moreover, by (5.14c)

$$q^{ol} = (D_{p^{ol}}\, q)_{p=1}$$

and hence for an arbitrary matrix ξ

$$q^{ol}_{ik}\, \xi_{ik} = [q^{ol}\, \xi^{*}] = (D_{p^{ol}}\, D_{\xi}\, U(p))_{p=1} = U^2(p^{ol},\xi)$$

$$= \lambda[p^{ol}][\xi] + \mu[p^{ol}(\xi + \xi^{*})].$$

Since this expression does not change if ξ is replaced by ξ^{*} the matrix q^{ol} must be symmetric:

(6.14) $\quad q^{ol}_{ik} = q^{ol}_{ki}$ for i,k = 1,2,3

We then have from (6.12)

(6.15) $\quad q^{ol}_{i3} = q^{ol}_{3i} = 0$ for i=1,2,3

(6.16) $\quad q^{ol}_{11,1} + q^{ol}_{12,2} = 0, \; q^{ol}_{21,1} + q^{ol}_{22,2} = 0, \; q^{ol}_{12} = q^{ol}_{21}$.

The general solution of equations (6.16) is of the form

(6.17) $\quad q^{ol}_{11} = \phi_{,22}\, , \; q^{ol}_{12} = q^{ol}_{21} = -\phi_{,12}\, , \; q^{ol}_{22} = \phi_{,11}$

with a suitable function $\phi(x_1,x_2)$. (Airy function).

We can show that ϕ has to satisfy the bi-harmonic equation.

For this purpose we write identity (6.13) component wise

(6.18) $\quad q^{ol}_{ik} = \lambda[p^{ol}]\, \delta_{ik} + \mu(p^{ol}_{ik} + p^{ol}_{ki}).$

We observe that by (6.17)

(6.19) $\quad \Delta^2\phi = q^{ol}_{11,22} - q^{ol}_{12,12} - q^{ol}_{21,21} + q^{ol}_{22,11}$

where Δ denotes the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

We have, quite generally, if ξ and ζ are matrices connected by

(6.20) $\qquad \zeta = (D_\xi\, q(p))_{p=1}$

or

$$\zeta_{ik} = \lambda\xi_{rr}\,\delta_{ik} + \mu(\xi_{ik} + \xi_{ki})$$

that

(6.21) $\qquad \zeta_{ik} = \zeta_{ki} \quad$ for all $i,k = 1,2,3$

and that

(6.22) $\qquad \zeta_{11,22} - \zeta_{12,12} - \zeta_{21,21} + \zeta_{22,11} - \frac{\lambda}{3\lambda+2\mu}\Delta(\zeta_{11}+\zeta_{22}+\zeta_{33})$

$$= 2\mu(\xi_{11,2} - \xi_{12,1})_{,2} - 2\mu(\xi_{21,2} - \xi_{22,1})_{1}$$

Applying this identity for $\xi = p^{01}$, $\zeta = q^{01}$, we find from (6.17), (6.15),(6.12) that

$$(1 - \frac{\lambda}{3\lambda+2\mu})\Delta^2\phi = 0$$

and hence, since λ and μ are positive,

(6.23) $\qquad \Delta^2\phi = 0.$

In addition, we have from (6.15),(6.18),(6.12) that

$$p^{01}_{31,2} = p^{01}_{32,1}\,,\; p^{01}_{13} + p^{01}_{31} = 0,\; p^{01}_{23} + p^{01}_{32} = 0.$$

Hence there exists a function $v = v(x_1,x_2)$ such that

(6.24) $\qquad p^{01}_{31} = p^{01}_{13} = v_{,1},\; p^{01}_{32} = -\,p^{01}_{23} = v_{,2}$

The scalar functions ϕ and v essentially describe the matrices p^{01}, q^{01}. We have already the bi-harmonic equation for ϕ. A differential equation for v follows from consideration of the terms of next order.

We have from (5.14b)

$$q^{10} = (D_{p^{10}}\, q(p))_{p=1}$$

or

(6.25a) $\qquad q^{10}_{ik} = \lambda\, p^{10}\,\delta_{ik} + \mu(p^{10}_{ik} + p^{10}_{ki}) \quad$ for $i,k = 1,2,3.$

Moreover, from (6.4),(6.5) for $\alpha=\beta=0$, using $q^{00} = 0$, $p^{00} = 1$:

(6.25b) $\qquad q^{10}_{i3} = p^{10}_{i1} = p^{10}_{i2} = 0$ for $i = 1,2,3.$

It follows immediately from (6.25a,b) that all elements

It follows immediately from (6.25a,b) that all elements p^{1o}_{ik} , q^{1o}_{ik} vanish

(6.26) $\quad p^{1o} = q^{1o} = 0.$

Then by (5.14d)

$$q^{2o} = (D_{p^{2o}}\, q)_{p=1}.$$

Since again from (6.4),(6.5) for $\alpha=1$, $\beta=0$ using (6.26)

$$q^{2o}_{i3} = p^{2o}_{i1} = p^{2o}_{i2} = 0 \quad \text{for } i=1,2,3$$

it follows in the same way that

(6.27) $\quad q^{2o} = p^{2o} = 0.$

Generally the assumptions $q^{oo}= 0$, $p^{oo}= 1$ imply that

(6.27a) $\quad q^{\alpha o} = p^{\alpha o} = 0 \quad \text{for } \alpha=1,2,3\ldots$

Equations (6.6),(6.7) for $\gamma=2$ then yield

(6.28) $\quad q^{o2}_{i3} = 0,\ q^{o2}_{i1,1} + q^{o2}_{i2,2} = 0 \quad \text{for } i=1,2,3.$

Moreover, by (5.14f)

(6.29) $\quad q^{o2} = (D_{p^{o1}}\, D_{p^{o1}}\, q + D_{p^{o2}}\, q)_{p=1}\ .$

Put

(6.30) $\quad Q^{o2} = (D_{p^{o1}}\, D_{p^{o1}}\, q)_{p=1}$

so that

(6.30a) $\quad q^{o2}_{ik} = \lambda[p^{o2}]\,\delta_{ik} + \mu(p^{o2}_{ik} + p^{o2}_{ki}) + Q^{o2}_{ik}\ .$

By (6.28)

$$(q^{o2}_{13} - q^{o2}_{31})_{,1} + (q^{o2}_{23} - q^{o2}_{32})_{,2} = 0$$

and hence, since $q^{o2} - Q^{o2}$ is symmetric,

(6.31) $\quad (Q^{o2}_{13} - Q^{o2}_{31})_{,1} + (Q^{o2}_{23} - Q^{o2}_{32})_{,2} = 0$

By (6.30)

$$\begin{aligned}(Q^{o2}_{ik} - Q^{o2}_{ki})\xi_{ik} &= [Q^{o2}(\xi^{*} - \xi)] \\ &= (D_{p^{o1}} D_{p^{o1}} D_{\xi}\ U(p) - D_{p^{o1}} D_{p^{o1}} D_{\xi^{*}} U(p))_{p=1} \\ &= U^{3}(p^{o1}, p^{o1}, \xi - \xi^{*})\end{aligned}$$

Since $\xi - \xi^*$ is skew symmetric, the terms with A,B,C, in the expression (3.15b) for $U^3(p^{o1},p^{o1},\xi - \xi^*)$ make no contribution, and we find

$$(Q^{o2}_{ik} - Q^{o2}_{ki})\xi_{ik} = 2\lambda[p^{o1}][p^{o1}(\xi^* - \xi)] + 2\mu[p^{o1}p^{o1}(\xi^* - \xi)]$$

identically in ξ. In particular

$$Q^{o2}_{3i} - Q^{o2}_{i3} = 2\lambda[p^{o1}](p^{o1}_{3i} - p^{o1}_{i3}) + 2\mu(p^{o1}_{3j}\, p^{o1}_{ji} - p^{o1}_{ij}\, p^{o1}_{j3})$$

For i=1,2 we obtain then, using (6.24),(6.13)

$$Q^{o2}_{3i} - Q^{o2}_{i3} = 4\lambda[p^{o1}]v_{,i} + 2\mu(v_{,1}(p^{o1}_{1i} + p^{o1}_{i1})$$
$$+ v_{,2}(p^{o1}_{2i} + p^{o1}_{i2})) + 4\mu v_{,i}p_{33}$$
$$= 2(q^{o1}_{1i}v_{,1} + q^{o1}_{2i}v_{,2} + q^{o1}_{33}v_{,i})$$

Hence by (6.15),(6.17)

$$Q^{o2}_{31} - Q^{o2}_{13} = 2(\phi_{,22}\, v_{,1} - \phi_{,12}\, v_{,2})$$

$$Q^{o2}_{32} - Q^{o2}_{23} = 2(-\phi_{,21}\, v_{,1} + \phi_{,11}\, v_{,2}).$$

Substituting these expressions into (6.31) we obtain the desired differential equation for v:

(6.32) $$v_{,11}\,\phi_{,22} - 2v_{,12}\,\phi_{,11} + v_{,22}\,\phi_{,11} = 0.$$

Unless ϕ vanishes identically the differential equations (6.23),(6.32) describe the behavior of tangential stresses and normal displacements to lowest order. Up to terms of order h^2 the stress matrix τ is given by hq^{o1} and thus by (6.15),(6.17)

$$\tau_{11} \approx h\phi_{,22},\quad \tau_{12} = \tau_{21} \approx - h\phi_{,12},\quad \tau_{22} \approx h\phi_{,11}$$

$$\tau_{i3} = \tau_{3i} \approx 0$$

where ϕ is a solution of the bi-harmonic equation. Thus the stresses normal to the surface are of order h^2 while the tangential stresses are of order h, unless the second derivatives of ϕ vanish. The differential equation for the tangential stresses does not involve the vertical deflection,since here for i=1,2

$$\frac{\partial x_3'}{\partial x_i} = p_{3i} = hp_{3i}^{ol} + O(h^2) = hv_{,i} + O(h^2)$$

the vertical deflection is given by

(6.33) $\quad x_3' = hv(x_1,x_2) + O(h^2)$,

where v satisfies equation (6.32). Thus up to terms of higher order

$$\tau_{11}\frac{\partial^2 x_3'}{\partial x_1^2} + 2\tau_{12}\frac{\partial^2 x_3'}{\partial x_1 \partial x_2} + \tau_{22}\frac{\partial^2 x_3'}{\partial x_2^2} = 0.$$

We next consider the case where all stress components q_{ik} are of the order h^2 i.e. where

(6.34) $\quad q^{ol} = 0.$

In this case the differential equation (6.32) collapses and higher order terms in the expansion have to be considered. It follows from (6.18) that

$$p_{11}^{ol} = p_{22}^{ol} = p_{33}^{ol} = p_{12}^{ol} + p_{21}^{ol} = 0.$$

Then also

$$p_{12,1}^{ol} = -\, p_{21,1}^{ol} = p_{11,2}^{ol} = 0,$$

$$p_{12,2}^{ol} = p_{21,2}^{ol} = -\, p_{22,1}^{ol} = 0.$$

Hence

$$p_{12}^{ol} = -p_{21}^{ol} = \text{const.}$$

By a suitable rigid motion in the x_1x_2 -plane one can bring about that this constant vanishes. That leaves only the components

$$p_{31}^{ol},\ p_{32}^{ol},\ p_{13}^{ol},\ p_{23}^{ol}$$

in p^{ol} which are described by the single function $v(x_1,x_2)$ through equations (6.24), where by (6.33) $hv(x_1,x_2)$ is essentially the normal displacement of the plate. We notice that now p^{ol} is skew symmetric:

(6.35) $\quad p^{ol} = -\, p^{ol^*}$.

Then by (6.30),(3.15b)

$$Q^{o2}_{ik} = [Q^{o2}\xi^*] = (D_{p^{o1}} D_{p^{o1}} D_\xi\, U(p))_{p=1}$$

$$= U^3(p^{o1}, p^{o1}, \varepsilon)$$

$$(6.36) \qquad = \lambda[\varepsilon][p^{o1*}p^{o1}] + \mu\, [(\xi + \varepsilon^*)\, p^{o1*}p^{o1}]$$

Substituting the specific values of the p^{o1}_{ik} in terms of v, we find

$$Q^{o2}_{ik} = 2\lambda(v_{,1}{}^2 + v_{,2}{}^2)\, \delta_{ik} + 2\mu\, v_{,i}\, v_{,k} \text{ for } i=k=1,2$$

$$(6.37) \qquad Q^{o2}_{i3} = Q^{o2}_{3i} = 0 \quad \text{for } i=1,2$$

$$Q^{o2}_{33} = (2\lambda + 2\mu)(v_{,1}{}^2 + v_{,2}{}^2).$$

Equations (6.37) show that Q^{o2} is symmetric. It follows from (6.30a) that also the matrix q^{o2} is symmetric. We have then from (6.28)

$$q^{o2}_{11,1} + q^{o2}_{12,2} = 0, \quad q^{o2}_{21,1} + q^{o2}_{22,2} = 0, \quad q^{o2}_{12} = q^{o2}_{21}$$

It follows from these equations that there exists an Airy function $\psi = \psi(x_1, x_2)$ such that

$$(6.38) \qquad q^{o2}_{11} = \psi_{,22} = , \quad q^{o2}_{12} = q^{o2}_{21} = \psi_{,12}, \quad q^{o2}_{22} = \psi_{,11}.$$

From (6.28) and the symmetry of q^{o2} we have for the remaining components of q^{o2}

$$(6.39) \qquad q^{o2}_{i3} = q^{o2}_{3i} = 0 \quad \text{for } i=1,2,3.$$

We obtain a differential equation for ψ by applying identity (6.22) to

$$\xi = p^{o2}, \quad \zeta = (D_{p^{o2}}\, q(p))_{p=1} = q^{o2} - Q^{o2}.$$

The right-hand side in (6.22) vanishes because of the compatibility conditions

$$(6.40) \qquad p^{o2}_{i2,1} = p^{o2}_{i1,2}$$

for p^{o2}; (see (6.5)). Substituting for the q^{o2}_{ik} and Q^{o2}_{ik} their values from (6.38),(6.39),(6.37) we obtain

$$(6.41) \qquad \Delta^2\, \psi = \frac{2\mu(3\lambda+2\mu)}{\lambda+\mu}\, (v_{,12}\, v_{,12} - v_{,11}\, v_{,22})$$

A second equation between ψ and v is derived from the fact that the relations connecting q^{o3} and p^{o3} have to be compatible. We find from (6.6),(6.7) for $\gamma=3$

$$q^{o3}_{i1} + 3q^{21}_{i1} = 0, \quad q^{o3}_{i1,1} + q^{o3}_{i2,2} + q^{21}_{i1,1} + {}^{21}_{i2,2} = 0$$

for i=1,2,3.

A combination of these equations is

$$(q^{o3}_{31} - q^{o3}_{13})_{,1} + (q^{o3}_{32} - q^{o3}_{23})_{,2}$$
$$= (3q^{21}_{13} - q^{21}_{31})_{,1} + (3q^{21}_{23} - q^{21}_{32})_{,2}$$

This relation will yield the desired differential equation. We find as in (5.14a,b,c,d,e,f)

(6.43a) $$q^{o3} = (D_{p^{o1}} D_{p^{o1}} D_{p^{o1}} q + 3D_{p^{o1}} D_{p^{o2}} q + D_{p^{o3}} q)_{p=1}$$

(6.43b) $$q^{21} = (D_{p^{1o}} D_{p^{1o}} D_{p^{1o}} q$$
$$+ D_{p^{o1}} D_{p^{2o}} q + 2D_{p^{1o}} D_{p^{11}} q + D_{p^{21}} q)_{p=1}$$

or for any matrix ξ

(6.44a) $$[q^{o3}\xi^*] = U^4(p^{o1},p^{o1},p^{o1},\xi)$$
$$+ 3U^3(p^{o1},p^{o2},\xi) + U^2(p^{o3},\xi)$$

(6.44b) $$[q^{21}\xi^*] = U^4(p^{1o},p^{1o},p^{o1},\xi) + U^3(p^{o1},p^{2o},\xi)$$
$$+2U^3(p^{1o},p^{11},\xi) + U^2(p^{21},\xi).$$

Since only combinations $q^{o3}_{ik} - q^{o3}_{ki}$ enter (6.42) we only need to determine

$$[(q^{o3} - q^{o3*})\xi^*] = [q^{o3}(\xi^* - \xi)]$$
$$= U^4(p^{o1},p^{o1},p^{o1},\xi - \xi^*) + 3U^3(p^{o1},p^{o2},\xi - \xi^*)$$
$$+ U^2(p^{o3},\xi - \xi^*).$$

Using the expressions (3.15a,b,c) for U^2,U^3,U^4 we see that only terms with coefficients λ or μ make contributions, since both p^{o1} and $\xi-\xi^*$ are skew symmetric. We obtain

$$(6.45)\qquad [(q^{o3} - q^{o3^*})\xi^*]$$
$$= - 6\lambda[p^{o1}p^{o1}][p^{o1}\xi^*] - 12\mu[p^{o1}p^{o1}p^{o1}\xi^*] + 6\lambda[p^{o2}][p^{o1}\xi^*]$$
$$+ 3\mu[(p^{o2}+ p^{o2^*})\, p^{o1}\xi^* + p^{o1}(p^{o2}+ p^{o2^*})\, \xi^*].$$

Here p^{o1} is the matrix given by

$$(6.45b)\qquad p^{o1} = \begin{pmatrix} 0 & 0 & -v_{,1} \\ 0 & 0 & -v_{,2} \\ v_{,1} & v_{,2} & 0 \end{pmatrix}$$

Taking for ξ the special matrix

$$\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

relation (6.45) reduces to

$$(6.46)\qquad q^{o3}_{31} - q^{o3}_{13} = + 12(\lambda+\mu)(v_{,1}{}^2+v_{,2}{}^2)v_{,1} + 6(\lambda+\mu)[p^{o2}]v_{,1}$$
$$- 6\mu p^{o2}_{22}\, v_{,1} + 3\mu(p^{o2}_{12} + p^{o2}_{21})v_{,2}$$

The p^{o2}_{ik} are connected with the known q^{o2}_{ik} by relations (6.30a) which yield

$$q^{o2} = (3\lambda + 2\mu)\, p^{o2} + Q^{o2}$$
$$q^{o2}_{22} = \lambda\, p^{o2} + 2\mu p^{o2}_{22} + Q^{o2}_{22}$$
$$q^{o2}_{12} = \mu(p^{o2}_{12} + p^{o2}_{21}) + Q^{o2}_{12}$$

Hence, using the values of the q^{o2}_{ik}, Q^{o2}_{ik} given by (6.37), (6.38), (6.39)

$$[p^{o2}] = - 2(v_{,1}{}^2+ v_{,2}{}^2) + \frac{1}{3\lambda+2\mu}\Delta\psi$$
$$2\mu\, p^{o2}_{22} = \psi_{,11} - \frac{\lambda}{3\lambda+2\mu}\Delta\psi - 2\mu\, v_{,2}{}^2$$
$$\mu(p^{o2}_{12} + p^{o2}_{21}) = \psi_{,12} - 2\mu v_{,1}v_{,2}$$

Substituting these values into (6.46) we obtain

$$q^{o3}_{31} - q^{o3}_{13} = 3(\psi_{,22}\, v_{,1} - \psi_{,12}\, v_{,2}).$$

Similarly, it follows that

$$q^{o3}_{32} - q^{o3}_{23} = 3(\psi_{,11}\, v_{,2} - \psi_{,12}\, v_{,1}).$$

This enables us finally to express the left hand side of (6.42) in terms of ψ and v:

(6.47) $$(q^{o3}_{31} - q^{o3}_{13})_{,1} + (q^{o3}_{32} - q^{o3}_{23})_{,2}$$

$$= 3(\psi_{,22}\, v_{,11} - 2\psi_{,12}\, v_{,12} + \psi_{,11}\, v_{,22})$$

It remains to carry out a similar reduction for the right hand side of (6.42). The matrix q^{21} is determined by (6.44b). Since by (6.26),(6.27) p^{1o} and p^{2o} vanish the terms with U^4 and U^3 in the expression (6.44b) make no contribution and we are left with

$$[q^{21}\xi^*] = U^2(p^{21},\xi)$$

or

$$q^{21}_{ik} = \lambda[p^{21}]\delta_{ik} + \mu(p^{21}_{ik} + p^{21}_{ki}).$$

Thus q^{21} is symmetric. Moreover by (6.4) for $\alpha=\beta=1$

$$q^{21}_{i3} = - q^{11}_{i1,1} - q^{11}_{i2,2}.$$

Hence

(6.48) $$3q^{21}_{13} - q^{21}_{31} = 2q^{21}_{13} = -2(q^{11}_{11,1} + q^{11}_{12,2}).$$

By (5.14e) for $p^{1o} = 0$

(6.49) $$q^{11} = (D_{p^{11}}\, q)_{p=1} = \lambda[p^{11}] + \mu(p^{11} + p^{11^*})$$

Here from (6.4) for $\alpha=0$, $\beta=1$

(6.50) $$q^{11}_{i3} = q^{o1}_{i1,1} - q^{o1}_{i2,2} = 0 \quad \text{for } i=1,2,3,$$

since $q^{o1} = 0$. In particular for i=3

(6.51) $$0 = q^{11}_{33} = \lambda[p^{11}] + 2\mu p^{11}_{33}\ .$$

On the other hand by (6.5),(6.45a)

(6.52) $$p^{11}_{ik} = p^{o1}_{i3,k} = -v_{,ik} \quad \text{for } i,k=1,2$$

$$[p^{11}] = -\Delta v + p^{11}_{33}$$

and thus by (6.51)

$$(6.53) \qquad [p^{11}] = \frac{-2\mu}{\lambda+2\mu} \Delta v.$$

Combining (6.49) with (6.52),(6.53) we get

$$q_{ik}^{11} = - \frac{2\lambda\mu}{\lambda+2\mu} \delta_{ik} \Delta v - 2\mu v_{,ik} \quad \text{for } i,k=1,2.$$

Substituting these values for the q_{ik}^{11} into (6.48) we find that

$$3q_{i3}^{21} - q_{3i}^{21} = \frac{+8\mu(\lambda+\mu)}{\lambda+2\mu} \Delta v_{,i} \quad \text{for } i=1,2$$

and hence that

$$(6.54) \qquad (3q_{13}^{21} - q_{31}^{21})_{,1} + (3q_{23}^{21} - q_{32}^{21})_{,2} = \frac{8\mu(\lambda+\mu)}{\lambda+2\mu} \Delta^2 v.$$

Combining (6.54),(6.47),(6.42) we obtain the desired second differential equation connecting v and ψ:

$$(6.55) \qquad \Delta^2 v = \frac{3(\lambda+2\mu)}{8\mu(\lambda+\mu)} (\psi_{,22} v_{,11} - 2\psi_{,12} v_{,12} + \psi_{,11} v_{,22})$$

Up to terms of higher order

$$w = hv(x_1, x_2)$$

represents the normal displacement of the plate, while

$$\tau_{11} = \frac{h^2}{2} q_{11}^{o2} = \frac{h^2}{2} \psi_{,22}, \quad \tau_{12} = - \frac{h^2}{2} \psi_{,12}, \quad \tau_{22} = \frac{h^2}{2} \psi_{,11}$$

represent the tangential components of the stress in the middle surface. These quantities represent at the same time the average tangential stresses throughout the thickness of the plate, since $q^{2o} = 0$ and the average of $q^{11} x_3 h$ vanishes. Equation (6.5) can be written in the form

$$(6.56) \qquad \Delta^2 w = \frac{2h}{N} (\tau_{11} w_{,11} + 2\tau_{12} w_{,12} + \tau_{22} w_{,22})$$

where

$$N = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)} h^3$$

is the plate stiffness constant. We introduce the Airy stress function

$$\chi = \frac{h^2}{2} \psi$$

so that

$$\tau_{11} = \chi_{,22}, \quad \tau_{12} = \tau_{21} = -\chi_{,12}, \quad \tau_{22} = \chi_{,11}.$$

Then equation (6.56) takes the form

$$\Delta^2 w = \frac{2h}{N}(\chi_{,22}\, w_{,11} - 2\chi_{,12}\, w_{,12} + \chi_{,11}\, w_{,22})$$

By (6.41) we have as second equation between χ and w

(6.58) $$\Delta^2 \chi = E(w_{,12}\, w_{,21} - w_{,22}\, w_{,11})$$

where

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

is Young's modulus, Equations (6.57),(6.58) represent the familiar v. Karman-Föppl system for buckling of plates.

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